Non-local Bose Gas as Microscopic origin of Schwarzschild black hole entropy

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Outlook

- Introduction
  - Third Law of Thermodynamics and its violation by BHs
  - Microscopic origin of the Bekenstein-Hawking entropy — previous considerations
- Microscopic origin of the Bekenstein-Hawking entropy via non-local gas models
  - entropy of **non-local gas** models with the ζ-function regularization
  - Examples
- Conclusion

Four Laws of Thermodynamics vs Four Laws of Black Hole mechanics

- There is a remarkable analogy between the laws of thermodynamics and the laws of black hole mechanics

Thermodynamics

- 0. $E, T, S, V, P, \ldots$
- 1. $dE = TdS - PdV$
- 2. $\delta S \geq 0$
- 3. $S \to 0$ if $T \to 0$

Black Hole mechanics
(Bardeen, Carter, Hawking,73'; Bekenstein 73')

- 0. surface gravity $\kappa = \frac{1}{M}$, $Q, a, \ldots$
- 1. $dM = \frac{1}{8\pi M} d\,\frac{\mathcal{A}}{4} + \ldots$
- 2. $\delta \mathcal{A} \geq 0$
- 3. States with $\kappa = 0$ are unattainable

- A missing link in this area is a precise statistical mechanical interpretation of entropies for all varieties of black holes.
- We can try to find a statistical mechanics model with the same dependence of entropy on other thermodynamic variables as a particular black hole has
- However, there is a problem with the third law of thermodynamics

Third Law of Thermodynamics

- In the Planck formulation : Entropy $S \to 0$ as $T \to 0$ $(\beta = \frac{1}{T} \to \infty)$

- In the Nernst formulation

$$\delta S(T,x) \equiv S(T,x) - S(T,x') \to 0 \quad as \quad T \to 0 \tag{1}$$

or

$$\lim_{T\to 0} S(T,x) - \text{universal constant}$$

- Unattainability of $T = 0$

REFS: W.Israel, 1986; R.Wald, 1997; F. Belgiorno and M. Martellini, 2004; C. Kehle and R. Unger, 2211.1574.

Violation of Third Law in BH Thermodynamics

- Schwarzschild black hole
  - Hawking temperature $T = \frac{1}{8\pi M}$
  - Bekenstein-Hawking entropy $S = \frac{1}{16\pi T^2} \to \infty \quad as \quad T \to 0$

Violation in Planck formulation

- Reissner-Nordstrom black hole
  - Hawking temperature $T = \frac{\sqrt{M^2 - Q^2}}{2\pi\left(\sqrt{M^2-Q^2}+M\right)^2} \to 0$ for $M \to Q$ or $M \to \infty$
  - BH entropy $S = \pi\left(\sqrt{M^2 - Q^2} + M\right)^2 \to \pi Q^2$ for $T \to 0$ **depends on** Q
- Kerr

Violation in Nernst formulation

Violation of Third Law in BH Thermodynamics
Few more examples

- dS
  - Hawking temperature $T = \frac{1}{2\pi\ell}$,
  - Bekenstein-Hawking entropy

    $$S = \frac{1}{4}\,\ell^2\,\Omega_{D-2} = \frac{1}{16\pi^2 T^2}\,\Omega_{D-2} \to \infty \quad as \quad T \to 0$$

- SAdS (in global coord.)
- RNAdS (in global and Poincare coords.)

BHs without Violation of Third Law

- SAdS in Poincare coords $ds^2 = \frac{r^2}{L^2}(-f dt^2 + d\vec{x}^2) + \frac{L^2}{r^2}\frac{dr^2}{f}$, $\quad f = 1 - \frac{r_h^{d+1}}{r^{d+1}}$
  - Hawking temperature $T = \dfrac{(d+1) r_h}{4\pi L^2}$
  - Bekenstein-Hawking entropy $(density)$ $S = \frac{1}{4G}\frac{r_h^d}{r^d} \to 0 \quad as \quad T \to 0$
- Deformed PAdS with BHs
  Example. Spacetime with topology $AdS_2 \times S^8$ Horowitz, Strominger 91

$$
\begin{aligned}
ds^2 &= c\,\rho^{9/20}\left[\frac{4}{25}\left(-\rho^2 h d\tau^2 + \frac{d\rho^2}{\rho^2 h}\right) + d\Omega_8^2\right] \\
h &= 1 - \left(\frac{\rho_0}{\rho}\right)^{\frac{14}{5}}, \quad T \sim \rho_0 \qquad (2) \\
S &\sim T^{9/5} \qquad (3)
\end{aligned}
$$

Physical systems with violation of the Third Law

- Lattice models.
  The question of whether the third law is satisfied can be decided completely in terms of ground-state degeneracies
  M. Aizenman, El. Lieb 80'

- Ice models.
  V. F. Petrenko and R. W. Whitworth, 99', *Physics of Ice*

- Strange metals.
  J. Zaanen et al. 15', *Holographic duality in condensed matter physics.*

Few Refs. on microscopic origin of BH entropy

- The problem of the microscopic origin of the Bekenstein-Hawking entropy of a black hole has attracted a lot of attention over the past 30 years.

  - **Wheeler** considered of the BH interior as "bag of gold" (**Almheiri et al 20**)

  - **Strominger and Vafa, 96'** $ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega_3^2$, $f(r) = \left(1 - (\frac{r_0}{r})^2\right)^2$, $r_0 = (\frac{8Q_H Q_F^2}{\pi^2})^{1/6}$, $S_{BH} = 2\pi\sqrt{\frac{Q_H Q_F^2}{2}}$
    D-0 branes interpretation: $d(n,c) \sim \exp(2\pi\sqrt{\frac{nc}{6}})$, $c = 6(\frac{1}{2}Q_F^2 + 1)$, $\quad n = Q_H$

$$S_{stat} = \ln d(Q_F, Q_H) \sim 2\pi\sqrt{Q_H(\frac{1}{2}Q_F^2 + 1)}$$

  - **'t Hooft 84'** proposed to relate BH entropy with the entropy of thermally excited quantum fields in the vicinity of the horizon.

  - Recent searches **Balasubramanian et al 22'** for internal geometries that provide the entropy of BH.

  - Matrix models corresponding to BH in spacetime with topology $AdS_2 \times S^8$, **Maldacena'23**

To summarize Introduction

- **Schwarzschild BHs violate 3-d law of thermodynamics.**
- **Schwarzschild BH entropies in D-dim $S \to \infty$ rather than zero when $T \to 0$.**
- **We search for quantum statistical models with such exotic thermodynamic behaviour.**

Free energy of non-local Bose gas (NLBG).

- d-dim non-local Bose gas (standard case $\alpha = 2$)

$$F_{BG}(d,\alpha) \quad = \quad \frac{\Omega_{d-1}}{\beta}\int_0^\infty \ln\left(1 - e^{-\lambda\,\beta\;k^\alpha}\right)\,k^{d-1}dk.$$

- Explicit form

$$F_{BG}(d,\alpha) = -\,\frac{2\pi^{d/2}}{d\Gamma(d/2)}\left(\frac{1}{\beta}\right)^{\frac{d}{\alpha}+1}\left(\frac{1}{\lambda}\right)^{\frac{d}{\alpha}}\Gamma\left(\frac{d}{\alpha}+1\right)\zeta\left(\frac{d}{\alpha}+1\right).$$

- Free energy of of D-dim Sch.BH $F_{BH}(D,\beta)$

- $F_{BH}(D,\beta) = F_{BG}(d,\beta)$

Reminder: Euler Gamma function

$$\Gamma(s) = \int_0^\infty e^{-x}\, x^{s-1}\, dx, \quad \Re s > 1 \tag{4}$$

$$\Gamma(n) = (n-1)! \tag{5}$$

$\Gamma(s)$ is a meromorphic function in $\mathbb{C}$ with simple poles at $s = 0, -1, -2, ...$

$\Gamma(s) \neq 0$ if $s \in \mathbb{C}$

Reminder: Riemann Zeta function

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{x^{s-1}dx}{e^x-1},\quad \Re s>1 \tag{6}$$

$\zeta(s)$ is a meromorphic function in $\mathbb{C}$ with simple a pole at $s=1$

- Zeros of $\zeta(s)$
  - "trivial zeros": $s=-2,-4,...$
  - "non-trivial zeros"in the strip $0<\Re s<1$
  - Riemann hypothesis: All non-trival zeros lie on line $\Re s=\frac{1}{2}$ (Hardy theorem)

Reminder: Functional Relations

$$\Gamma(s)\,\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$

$$\Gamma(s)\,\zeta(s) = \frac{\pi^s\,\zeta(1-s)}{2^{1-s}\sin(\frac{\pi(1-s)}{2})},$$

D=4 Schwarzschild BH vs Bose Gas. 1/3

- Schwarzschild solution

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2,$$

- Hawking temperature and Bekenstein-Hawking entropy

$$T = \frac{1}{8\pi M}, \qquad S = 4\pi M^2 = \frac{\beta^2}{16\pi}$$

- Free energy

$$F = \frac{\beta}{16\pi}$$

- Equalizing: $F_{BG}(\beta) = F_{BH}(\beta)$

$$-\frac{\pi^{d/2}}{\beta^{\frac{d}{2}+1}\lambda^{\frac{d}{2}}}\,\zeta\left(\frac{d}{2}+1\right) = \frac{\beta}{16\pi} \quad (*)$$

- To fulfill (*) we have to assume

$$d = -\,4, \qquad \lambda^2 = -\,\frac{\pi}{16\,\zeta(-1)}.$$

- Taking into account that $\zeta(-1) = -1/12$, we get

$$\lambda \;=\; \sqrt{\frac{3\pi}{4}},$$

D=4 Schwarzschild BH vs Bose Gas. 3/3

- Therefore, we obtain that the thermodynamics of the 4-dim Schwarzschild BH is equivalent to the thermodynamics of the Bose gas in $d = -4$ spatial dimensions.

- We understand the thermodynamics of the Bose gas in negative spatial dimensions in the sense of the analytical continuation of the right hand site of

$$F_{BG} = -\frac{\pi^{d/2}}{\beta^{\frac{d}{2}+1}\lambda^{\frac{d}{2}}}\zeta\left(\frac{d}{2}+1\right).$$

D>4 Schwarzschild BH vs Bose Gas

- D-dimensional Schwarzschild black hole, $D \geq 4$,

$$ds^2 = -\left(1 - \frac{{r_h}^{D-3}}{r^{D-3}}\right) dt^2 + \frac{dr^2}{1 - \frac{{r_h}^{D-3}}{r^{D-3}}} + r^2 d\omega_{D-2}^2,$$

- Hawking temperature $T = 1/\beta = \frac{D-3}{4\pi r_h}$
  r_h is the radius of the horizon.
- The entropy and the free energy are

$$S = \frac{\Omega_{D-2}}{4}\left(\frac{D-3}{4\pi}\,\frac{1}{T}\right)^{D-2};\ F = \frac{(D-3)^{D-3}\beta^{D-3}\,\Omega_{D-2}}{4(4\pi)^{D-2}}$$

$S \to \infty$, when $T \to 0$ — a violation of the 3-d law

D>4 Schwarzschild BH vs Bose Gas.

- Equalizing: $F_{BG}(\beta) = F_{BH}(\beta)$ we get

$$-\frac{\pi^{d/2}}{\Gamma(d/2+1)}\left(\frac{1}{\beta}\right)^{\frac{d}{\alpha}+1}\left(\frac{1}{\lambda}\right)^{\frac{d}{\alpha}}\Gamma\left(\frac{d}{\alpha}+1\right)\zeta\left(\frac{d}{\alpha}+1\right)$$

$$=\frac{(D-3)^{D-3}}{4(4\pi)^{D-2}}\,\beta^{D-3}\,\frac{2\pi^{\frac{D-1}{2}}}{\Gamma(\frac{D-1}{2})}$$

- powers of β: $d = -(D-2)\,\alpha$,

$$F_{BG} = -\frac{\pi^{-\frac{(D-2)\alpha}{2}}}{\Gamma(1-\frac{(D-2)\alpha}{2})}\,\beta^{D-1}\,\lambda^{D-2}\Gamma\,(3-D)\,\zeta\,(3-D)\,.$$

D>4 Schwarzschild BH vs Bose Gas.
Necessary & sufficient conditions

- Since $F_{BH} > 0$, using the functional relations

$$
\begin{aligned}
i) \quad \Gamma(3-D)\,\zeta(3-D) \quad &= \quad \frac{\zeta(D-2)}{2^{D-2}\pi^{D-3}\sin(\frac{\pi(D-2)}{2})} \\
ii) \quad \Gamma\left(\frac{(D-2)\alpha}{2}\right)\Gamma\left(1-\frac{(D-2)\alpha}{2}\right) \quad &= \quad \frac{\pi}{\sin(\pi\frac{(D-2)\alpha}{2})}
\end{aligned}
$$

- we obtain **necessary & sufficient** conditions for the existence of λ, $0<\lambda<\infty$, solving $F_{BG} = F_{BH}$

$$
0 < -\frac{\sin(\frac{\pi(D-2)}{2})}{\Gamma(\frac{(D-2)\alpha}{2})\sin(\pi\frac{(D-2)\alpha}{2})} < \infty \qquad (**)
$$

D>4 Schwarzschild BH vs Bose Gas.
Solutions to inequalities (**)

$$0 < -\frac{\sin(\frac{\pi(D-2)}{2})}{\Gamma(\frac{(D-2)\alpha}{2})\sin(\pi\frac{(D-2)\alpha}{2})} < \infty \qquad (**)$$

- Since D is a natural number, $\sin(\frac{\pi(D-2)}{2})$ takes three values:

$$\sin(\frac{\pi(D-2)}{2}) = \begin{cases} 1 & \text{for} \quad D = 4k+3, \qquad k = 1,2,3,... \\ 0 & \text{for} \quad D = 2k, \qquad k = 2,3,4,... \\ -1 & \text{for} \quad D = 4k+1, \qquad k = 1,2,3,... \end{cases}. \qquad (7)$$

D>4 Schwarzschild BH vs Bose Gas.
First series, $D = 2k$

$$0 < -\frac{\sin(\frac{\pi(D-2)}{2})}{\Gamma(\frac{(D-2)\alpha}{2})\sin(\pi\frac{(D-2)\alpha}{2})} < \infty \qquad (**)$$

- For $D = 2k$ — the zeros of the sines must be compensated: $\alpha = \frac{p}{k-1}$, $p = 1, 2, ...$

Regularization $D = 2k + \epsilon$,

$$K[k,p,\epsilon] \equiv -\frac{\sin(\frac{\pi(D-2)}{2})}{\sin(\pi\frac{(D-2)\alpha}{2})}\Big|_{D=2k+\epsilon,\alpha=\frac{p}{k-1}}, \quad \lim_{\epsilon\to 0} K[k,p,\epsilon] = \frac{(-1)^k(k-1)}{2p}$$

$$k = 2, 3, 4, ...$$

D>4 Schwarzschild BH vs Bose Gas.
4 series of solutions

- 4 series of solutions

D	d	α
$D = 4k+1, \quad k = 1,2,3...$	$d = (4k-1)\lvert\alpha\rvert$	$\alpha = -q,\ q = 1,2,3$
$D = 4k+1, \quad k = 1,2,3...$	$d = -(4k-1)\alpha$	$\frac{4r}{4k-1} < \alpha < \frac{2(2r+1)}{4k-1}, \quad r = 0,1,2,...$
$D = 4k+3, \quad k = 1,2,3...$	$d = -(4k+1)\alpha$	$\frac{2(2r+1)}{4k+1} < \alpha < \frac{4(r+1)}{4k+1}, \quad r = 0,1,2...$
$D = 2k, \quad k = 2,3,4...$	$d = -2(k-1)\alpha$	$\alpha = \frac{p}{k-1}, \quad p = 1,2,..$

- Euclid $d = 3$
  Kaluza-Klein $d = 5$
  Superstrings $d = 10$
  Here $d < 0$

De Sitter and Bose Gas

$$ds^2 = -\left(1 - \frac{r^2}{\ell^2}\right) dt^2 + \left(1 - \frac{r^2}{\ell^2}\right)^{-1} dr^2 + r^2 d\omega_{D-2}^2,$$

$$T = \frac{1}{2\pi\ell}, \quad S = \frac{1}{16\pi^2 T^2}\,\Omega_{D-2}, \quad F = \frac{\beta}{16\pi}\,\Omega_{D-2},\, D \geq 3$$

- $F_{dS} = F_{BG}$ **gives**

$$\alpha = 2k,\, d = -4k,$$

$$\lambda^2 = \frac{1}{(2k)!}\,\frac{3\pi^{\frac{D-3}{2}+2k}}{\Gamma(\frac{D-1}{2})},\, k = 1, 2, 3, ...\, D = 4, 5, ...$$

RN Extremal BH

D=4, We know that entropy for ERN at zero temperature

$$S = \pi Q^2 \tag{8}$$

For Bose gas

$$F_{BG} = -\frac{2\pi^{d/2}}{(2\pi)^d \, d\Gamma(d/2)} \left(\frac{1}{\beta}\right)^{\frac{d}{\alpha}+1} \left(\frac{1}{\lambda}\right)^{\frac{d}{\alpha}} \Gamma\left(\frac{d}{\alpha}+1\right) \zeta\left(\frac{d}{\alpha}+1\right). \tag{9}$$

and entropy

$$E_{BG} = \beta^2 \frac{\partial F_{BG}}{\partial \beta} = (\frac{d}{\alpha}+1)\beta^{-\frac{d}{\alpha}} \frac{\pi^{d/2}}{\Gamma(d/2+1)} \left(\frac{1}{\lambda}\right)^{\frac{d}{\alpha}} \Gamma\left(\frac{d}{\alpha}+1\right) \zeta\left(\frac{d}{\alpha}+1\right) \tag{10}$$

We take $\alpha = \infty$ and we get

$$\pi Q^2 = \frac{\pi^{d/2}}{\Gamma(d/2+1)} \Gamma(1) \, \zeta(1) \tag{11}$$

Extremal RNBH. D=4

Let us denote $d/2+1 \equiv -n, \quad n=0,1,2$, and we left with

$$\frac{\pi Q^2}{\pi^{n+1}} = \frac{\zeta(1)}{\Gamma(-n)} \qquad (\#)$$

We understand the RHS of (#) as $\lim_{\epsilon\to 0} \frac{\zeta(1+\epsilon)}{\Gamma(-n+\epsilon)} = (-1)^n \frac{1}{n!} \Rightarrow n = 2k$

$$Q^2 \equiv M^2 = \frac{\pi^{2k}}{(2k)!}, \quad k=1,2,3$$

In the usual unit and correct multiple $(2\pi)^d = (2\pi)^{-2(n-1)}$

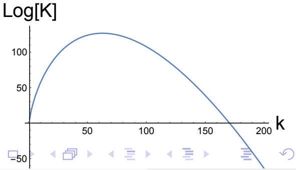
$$M^2 = K(k) M_{pl}^2, \quad K(k) = \frac{\pi^{2k}(2\pi)^{2(2k-1)}}{(2k)!}, \qquad k=1,2,3 \qquad (12)$$

We get quantization of the mass of the BH!

Using the Stirling formula $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

we get $M^2 \approx \frac{1}{8\pi^{5/2}\sqrt{k}}\left(\frac{2e\pi^3}{k}\right)^{2k} M_{pl}^2$

$2e\pi^3 \approx 168$



Extremal RNBHs. D>4

For the extremal case we have $T = 0$ and entropy

$$S_{ext} = \frac{1}{4}\Omega_{D-2}M^{\frac{D-2}{D-3}}, \qquad \Omega_{D-2} = \frac{2\pi^{(D-1)/2}}{\Gamma((D-1)/2)}$$

$$S_{BG} = S_{BG}$$

$$\frac{1}{4}\frac{2\pi^{(D-1)/2}}{\Gamma((D-1)/2)}M^{\frac{D-2}{D-3}} = (\frac{d}{\alpha}+1)\beta^{-\frac{d}{\alpha}}\frac{\pi^{d/2}}{\Gamma(d/2+1)}\left(\frac{1}{\lambda}\right)^{\frac{d}{\alpha}}\Gamma\left(\frac{d}{\alpha}+1\right)\zeta\left(\frac{d}{\alpha}+1\right)$$

We take $\alpha = \infty$ and get

$$\frac{1}{4}\frac{2\pi^{(D-1)/2}}{\Gamma((D-1)/2)}M^{\frac{D-2}{D-3}} = \frac{\pi^{d/2}}{\Gamma(d/2+1)}\Gamma(1)\,\zeta(1) \qquad (13)$$

Let us denote $d/2+1 \equiv -n, \quad n = 0, 1, 2,$ and we left with

$$\frac{1}{4\pi^{n+1}}\frac{2\pi^{(D-1)/2}}{\Gamma((D-1)/2)}M^{\frac{D-2}{D-3}} = (-1)^n\frac{1}{n!} \Rightarrow n = 2k \qquad (14)$$

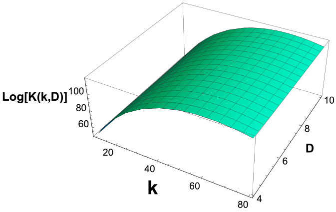
RN Extremal BHs. D>4

In the usual unit and correct multiple $(2\pi)^d = (2\pi)^{-2(n-1)}$ (the same as it was for D=4)

$$\left(\frac{M}{M_{pl}}\right)^{\frac{D-2}{D-3}} = K(D,k),$$

$$K(D,k) = \Gamma(\frac{D-1}{2})\frac{4\pi^{2k+1}(2\pi)^{2(2k-1)}}{2\pi^{(D-1)/2}(2k)!}$$

$$k = 1, 2, 3, ...$$



Very weak dependence on D

- NLBG with $\alpha = \infty$, $d = -2(2k+1)$ recovers the entropy of extremal RNBH at zero T.

Conclusion

- The model of Bose gas in negative dimension recovers the temperature dependence of entropy of the D=4 Schwarzschild black hole
- We use the Riemann zeta function analytical continuation to define the entropy of the Bose gas in negative dimension.
- $D > 4$, in particular, for $D = 4k + 1$, $d = (4k - 1)q$, $\alpha = -q$; $k, q = 1, 2, 3, ...$
- Bose gas with $\alpha = 2k$ in negative dimension $d = -4k$ recovers the temperature dependence of entropy of the dS spacetime (D -arbitrary).
- Quantization of the mass of the extremal RN BH.