(Quantum) Gravity from Yang-Mills Theory

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- joint work on *double copy & double field theory* with:
R. Bonezzi, C. Chiaffrino, F. Diaz-Jaramillo, J. Plefka,
2109.01153, 2203.07397, 2212.04513 [hep-th]

- joint work with A. Pinto on *cosmological double field theory*,
2207.14788 [hep-th]

- with R. Bonezzi, C. Chiaffrino, F. Diaz-Jaramillo,
work in progress

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Motivation & Outline

- Inspired by *double copy* from scattering amplitudes, slogan: Gravity = (Yang-Mills)^2 but off-shell?
  [Bern, Carrasco & Johansson (2008), Kawai-Lewellen-Tye (KLT) relations (1986)]

- Lagrangian double copy Replacing color indices as $a \to \bar{\mu}$, corresponding to a *second* set of spacetime momenta $\bar{k}$:

$$A_\mu{}^a(k) \;\to\; e_{\mu\bar{\mu}}(k,\bar{k})$$

  yields gravity as a double field theory (DFT) to cubic order.

- Algebraic double copy (gauge invariant and off-shell)
  $\to$ Yang-Mills theory as L_∞-algebra $\mathcal{K}\otimes\mathfrak{g}$
  $\to$ DFT as L_∞-algebra on $\mathcal{K}\otimes\bar{\mathcal{K}}$

- Weakly constrained theory (to quartic order!) on tori,
  $\to$ field theory of momentum and genuine winding modes
  $\to$ time-dependent (cosmological) backgrounds?

Part I: Lagrangian Double Copy

Yang-Mills at Quadratic Order

Yang-Mills action in D dimensions,

$$S_{\mathsf{YM}} = -\frac{1}{4}\int d^D x\, \kappa_{ab}\, F^{\mu\nu\, a} F_{\mu\nu}{}^b$$

expanded to quadratic order in momentum space:

$$S^{(2)}_{\mathsf{YM}} = -\frac{1}{2}\int_k \kappa_{ab}\, k^2\, \Pi^{\mu\nu}(k) A_\mu{}^a(-k) A_\nu{}^b(k) \ ,$$

with $\int_k := \int d^D k$ and the projector $\quad [\, \Pi^{\mu\rho}\Pi_{\rho\nu} = \Pi^\mu{}_\nu \,]$

$$\Pi^{\mu\nu}(k) \equiv \eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}$$

Gauge invariance

$$\delta A_\mu{}^a(k) = k_\mu \lambda^a(k)\,, \qquad \Pi^{\mu\nu}(k) k_\nu \equiv 0 \ ,$$

where the gauge parameter $\lambda^a(k)$ is an arbitrary function.

Lagrangian Double Copy at Quadratic Order

Substitution $A_\mu{}^a(k) \;\to\; e_{\mu\bar{\mu}}(k,\bar{k}) \equiv e_{\mu\bar{\mu}}(K)$ together with

$$\kappa_{ab} \;\to\; \bar{\Pi}^{\bar{\mu}\bar{\nu}}(\bar{k})$$

yields

$$S^{(2)}_{\text{DC}} = -\frac{1}{4}\int_{k,\bar{k}} k^2\,\Pi^{\mu\nu}(k)\bar{\Pi}^{\bar{\mu}\bar{\nu}}(\bar{k})e_{\mu\bar{\mu}}(-K)e_{\nu\bar{\nu}}(K)$$

Democratic between k and $\bar{k}$ due to *level-matching constraint*

$$k^2 = \bar{k}^2$$

Gauge invariant under

$$\delta e_{\mu\bar{\mu}} \;=\; k_\mu\,\bar{\lambda}_{\bar{\mu}} + \bar{k}_{\bar{\mu}}\,\lambda_\mu\;,$$

with *two* independent gauge parameters λ_μ and $\bar{\lambda}_{\bar{\mu}}$.

Double Field Theory at Quadratic Order

Claim: This *is* double field theory to quadratic order.

To see this we introduce 'auxiliary' scalar ϕ (DFT dilaton):

$$S^{(2)}_{\text{DC}} \propto \int_{k,\bar{k}} \Big(k^2 e^{\mu\bar{\nu}} e_{\mu\bar{\nu}} - (k^\mu e_{\mu\bar{\nu}})^2 - (\bar{k}^{\bar{\nu}} e_{\mu\bar{\nu}})^2 - k^2\phi^2 + 2\phi\, k^\mu \bar{k}^{\bar{\nu}} e_{\mu\bar{\nu}} \Big)$$

so that integrating out ϕ,

$$\phi = \frac{1}{k^2} k^\mu \bar{k}^{\bar{\nu}} e_{\mu\bar{\nu}} \qquad (\Rightarrow\ \delta\phi = k \cdot \lambda + \bar{k} \cdot \bar{\lambda}\ )$$

we recover the double copy action above.

In (doubled) position space this is (free) DFT with derivatives $\partial_\mu = \frac{\partial}{\partial x^\mu}$ and $\bar{\partial}_{\bar{\mu}} = \frac{\partial}{\partial \bar{x}^{\bar{\mu}}}$ subject to "weak constraint"

$$\Box \equiv \partial^\mu \partial_\mu = \bar{\partial}^{\bar{\mu}} \bar{\partial}_{\bar{\mu}}\,.$$

For $\partial_\mu = \bar{\partial}_{\bar{\mu}}$: (linearized) gravity + B-field + dilaton.

[Siegel (1993), Hull & Zwiebach (2009)]

Lagrangian Double Copy at Cubic Order

Cubic part of Yang-Mills action

$$S^{(3)}_{\mathsf{YM}} = -\mathsf{g}_{\mathsf{YM}} \int d^D x\, f_{abc}\, \partial^\mu A^{\nu a}\, A_\mu{}^b\, A_\nu{}^c$$

reads in momentum space

$$S^{(3)}_{\mathsf{YM}} \;\propto\; \mathsf{g}_{\mathsf{YM}} \int_{k_1,k_2,k_3} \delta(k_1{+}k_2{+}k_3)\, f_{abc}\, \Pi^{\mu\nu\rho}(k_1,k_2,k_3)\, A_{1\mu}{}^a A_{2\nu}{}^b A_{3\rho}{}^c$$

with short-hand notation $A_i \equiv A(k_i)$ and

$$\Pi^{\mu\nu\rho}(k_1,k_2,k_3) \equiv \eta^{\mu\nu} k_{12}^\rho + \eta^{\nu\rho} k_{23}^\mu + \eta^{\rho\mu} k_{31}^\nu \qquad [\, k_{ij} \equiv k_i - k_j \,]$$

$\rightarrow$ obvious double copy rule:

$$f_{abc} \;\rightarrow\; \bar{\Pi}^{\bar\mu\bar\nu\bar\rho}(\bar{k}_1,\bar{k}_2,\bar{k}_3)$$

Double Field Theory at Cubic Order

Double copied cubic action (with $K = (k, \bar{k})$ and $e_i = e(K_i)$)

$$S^{(3)}_{\mathsf{DC}} \propto \int dK_{123}\, \delta(K_1 + K_2 + K_3)$$
$$\times\, \Pi^{\mu\nu\rho}(k_1, k_2, k_3)\, \bar{\Pi}^{\bar{\mu}\bar{\nu}\bar{\rho}}(\bar{k}_1, \bar{k}_2, \bar{k}_3)\, e_{1\,\mu\bar{\mu}}\, e_{2\,\nu\bar{\nu}}\, e_{3\,\rho\bar{\rho}}$$

yields in (doubled) position space

$$S^{(3)}_{\mathsf{DC}} \propto \int dx d\bar{x}\; e_{\mu\bar{\mu}} \Big[2\partial^{\mu} e_{\rho\bar{\rho}}\, \bar{\partial}^{\bar{\mu}} e^{\rho\bar{\rho}} - 2\partial^{\mu} e_{\nu\bar{\rho}}\, \bar{\partial}^{\bar{\rho}} e^{\nu\bar{\mu}} - 2\partial^{\rho} e^{\mu\bar{\rho}}\, \bar{\partial}^{\bar{\mu}} e_{\rho\bar{\rho}}$$
$$+ \partial^{\rho} e_{\rho\bar{\rho}}\, \bar{\partial}^{\bar{\rho}} e^{\mu\bar{\mu}} + \bar{\partial}_{\bar{\rho}} e^{\mu\bar{\rho}}\, \partial_{\rho} e^{\rho\bar{\mu}} \Big]$$

This is cubic double field theory in Siegel gauge (which implies de Donder gauge for vanishing dilaton) !

Part II: Algebraic Double Copy

L_∞-algebra and Perturbative Field Theory

Algebraic interpretation of any (perturbative) field theory

[Zwiebach (1993), O.H. & Zwiebach (2017)]

$$S = \frac{1}{2}\Big\langle A\,, B_1(A)\Big\rangle + \frac{1}{3!}\Big\langle A\,, B_2(A,A)\Big\rangle + \frac{1}{4!}\Big\langle A\,, B_3(A,A,A)\Big\rangle + \cdots$$

on integer graded vector space

$$\mathcal{X} = \bigoplus_i X_i$$

structure: graded symmetric maps $B_1, B_2, B_3, \ldots$, $|B_n| = 1$, obeying generalized Jacobi identities:

- B_1 is nil-potent differential: $B_1^2 \ = \ 0$
- $B_1(B_2(x,y)) + B_2(B_1(x),y) + (-1)^x B_2(x,B_1(y)) \ = \ 0$
- Jacobi up to homotopy

$$B_2(B_2(x,y),z) + (-1)^{yz} B_2(B_2(x,z),y) + (-1)^x B_2(x,B_2(y,z))$$
$$+\, B_1(B_3(x,y,z)) + B_3(B_1(x),y,z) + \text{two terms} \ = \ 0$$

L_∞-algebra of Yang-Mills Theory I

Formulation of Yang-Mills suggested by string field theory:

$$S = \int d^D x \, \mathrm{Tr}\left\{\tfrac{1}{2}\, A^\mu \Box A_\mu - \tfrac{1}{2}\,\varphi^2 + \varphi\,\partial_\mu A^\mu - \partial_\mu A_\nu\,[A^\mu, A^\nu] - \tfrac{1}{4}\,[A^\mu, A^\nu]\,[A_\mu, A_\nu]\right\}$$

with Lie algebra valued fields, $A_\mu = A_\mu^a\, t_a$, etc.

Algebra: gauge parameters, fields, EoM & Noether identities:

$$\begin{array}{ccccccc} X_{-1} & \xrightarrow{B_1} & X_0 & \xrightarrow{B_1} & X_1 & \xrightarrow{B_1} & X_2 \\ \lambda & & \mathcal{A} & & \mathcal{E} & & \mathcal{N} \end{array}$$

with doublets $\mathcal{A} = (A^\mu, \varphi)$ and $\mathcal{E} = (E^\mu, E)$.

Linearized gauge invariance/EoM: $\delta\mathcal{A} = B_1(\lambda)$ and $B_1(\mathcal{A}) = 0$:

$$B_1(\lambda) = \begin{pmatrix} \partial^\mu \lambda \\ \Box \lambda \end{pmatrix} \in X_0\,, \qquad B_1(\mathcal{A}) = \begin{pmatrix} \Box A^\mu - \partial^\mu \varphi \\ \partial \cdot A - \varphi \end{pmatrix} \in X_1$$

L_∞-algebra of Yang-Mills Theory II

Non-linear Yang-Mills equations

$$B_1(\mathcal{A}) + \tfrac{1}{2}\, B_2(\mathcal{A},\mathcal{A}) + \tfrac{1}{3!}\, B_3(\mathcal{A},\mathcal{A},\mathcal{A}) = 0$$

with, e.g.,

$$\begin{aligned} B_2^\mu(A_1,A_2) &= \partial_\nu[A_1^\nu, A_2^\mu] + [\partial^\mu A_1^\nu - \partial^\nu A_1^\mu, A_{2\nu}] + (1 \leftrightarrow 2) \\ B_3^\mu(A_1,A_2,A_3) &= [A_{\nu 1}, [A_2^\nu, A_3^\mu]] + 5 \text{ more terms} \end{aligned}$$

Non-linear gauge transformations & algebra

$$\delta\mathcal{A} = B_1(\lambda) + B_2(\lambda,\mathcal{A})$$

where

$$B_2(\lambda,\mathcal{A}) = \begin{pmatrix} [A^\mu,\lambda] \\ \partial_\nu[A^\nu,\lambda] \end{pmatrix} , \qquad B_2(\lambda_1,\lambda_2) = -[\lambda_1,\lambda_2]$$

Classical consistency $\Leftrightarrow$ generalized Jacobi identities

The Kinematic Algebra: Stripping Off Color

Realize L_∞-algebra on $\mathcal{X}$ as tensor product:

[A. M. Zeitlin (2010)]

$$\mathcal{X} = \mathcal{K} \otimes \mathfrak{g}$$

$\mathfrak{g}$-valued field lives in tensor product, $x = x^a t_a \to x^a \otimes t_a$

L_∞ structure comes from algebraic structure on $\mathcal{K}$:

$$B_1(x) = m_1(x^a) \otimes t_a$$

$$B_2(x_1, x_2) = \pm f^a{}_{bc}\, m_2(x_1^b, x_2^c) \otimes t_a$$

$$B_3(A_1, A_2, A_3) = 2\, f^a{}_{be} f^e{}_{cd}\, m_3(A^b_{(1}, A^c_2, A^d_{3)}) \otimes t_a$$

Specifically, m_1, m_2, m_3 form C_∞-algebra (homotopy version of commutative and associative differential graded algebra)

$$m_2(m_2(u_1,u_2),u_3) - m_2(u_1, m_2(u_2,u_3)) = m_1(m_3(u_1,u_2,u_3)) + m_3(m_1(u_1),u_2,u_3) + \text{two terms}$$

More on Kinematic Algebra

Kinematic vector space $\mathcal{K} = \bigoplus_{i=0}^{3} K_i$ admits $\mathbb{Z}_2$ grading into $\mathcal{K}^{(0)} \oplus \mathcal{K}^{(1)}$:

$$
\begin{array}{ccccccc}
K_0 & \xrightarrow{m_1} & K_1 & \xrightarrow{m_1} & K_2 & \xrightarrow{m_1} & K_3 \\
\lambda & & A_\mu & & E & & \\
 & & \varphi & & E_\mu & & \mathcal{N}
\end{array}
$$

$\Rightarrow$ second nilpotent operator b with $|b| = -1$, acting, e.g., as

$$
b(\mathcal{N}) = \begin{pmatrix} \mathcal{N} \\ 0 \end{pmatrix} \in K_2\,, \quad b\begin{pmatrix} E \\ E_\mu \end{pmatrix} = \begin{pmatrix} E_\mu \\ 0 \end{pmatrix} \in K_1\,, \quad b\begin{pmatrix} A_\mu \\ \varphi \end{pmatrix} = \varphi \in K_0\,,
$$

satisfying

$$
b^2 = 0\,, \qquad m_1 b + b m_1 = \Box\,\mathbf{1}
$$

Hidden 'Lie-type' algebra on $\mathcal{K}$

$\mathcal{K}$ born as an 'associative-type' algebra, but is also of 'Lie-type': failure of b to obey Leibniz w.r.t. m_2 almost Lie bracket :

$$b_2(u_1, u_2) := bm_2(u_1, u_2) - m_2(bu_1, u_2) - (-1)^{u_1} m_2(u_1, bu_2)$$

Batalin-Vilkovisky (BV) algebra:

commutative, associative product m_2 and differential b, $b^2 = 0$, of *second order* $\Rightarrow$ b_2 obeys Jacobi & compatibility with m_2.

$\mathcal{K}$ fails to be BV algebra because m_2 is not associative

$\rightarrow$ BV_∞ algebra? Not quite, since $m_1 b + b m_1 = \Box\mathbf{1}$

$\rightarrow$ $\mathrm{BV}_\infty^\Box$ algebra, where, e.g.,

$$m_1(b_2(u_1, u_2)) + b_2(m_1(u_1), u_2) + (-1)^{u_1} b_2(u_1, m_1(u_2)) = 2\, m_2(\partial^\mu u_1, \partial_\mu u_2) \neq 0$$

[Reiterer, 1912.03110 [math-ph]]

Double Field Theory = Double Copy of Yang-Mills I

Introduce second copy $\bar{\mathcal{K}}$ of kinematic algebra

$\Rightarrow$ tensor product $\mathcal{K}\otimes\bar{\mathcal{K}}$ is chain complex of functions of $(x,\bar{x})$

two natural differentials of opposite degrees:

$$B_1 := m_1\otimes\mathbf{1}+\mathbf{1}\otimes\bar{m}_1 \qquad b^- := \tfrac{1}{2}(b\otimes\mathbf{1}-\mathbf{1}\otimes\bar{b})$$

satisfying $B_1^2=0$, $(b^-)^2=0$ and

$$B_1 b^- + b^- B_1 = \Delta\,, \qquad \Delta := \tfrac{1}{2}(\Box-\bar{\Box})\,.$$

Eliminate 'Δ-failure' by going to subspace

$$\mathcal{V}_{\mathsf{DFT}} := \left\{\psi\in\mathcal{K}\otimes\bar{\mathcal{K}}\,\middle|\,\Delta\psi=0\,,\ b^-\psi=0\right\}$$

$\Rightarrow$ precisely complex of DFT! E.g. for fields

$$(e_{\mu\bar{\nu}}\,,\ e\,,\ \bar{e}\,,\ f_\mu\,,\ \bar{f}_{\bar{\mu}})\ \in\ (K_1\otimes\bar{K}_1)\oplus(K_0\otimes\bar{K}_2)\oplus(K_2\otimes\bar{K}_0)$$

includes graviton, B-field and scalar dilaton.

Double Field Theory = Double Copy of Yang-Mills II

Non-linear structure? $\rightarrow$ higher brackets like 2-bracket:

$$B_2 := -\tfrac{1}{4}\left(b_2 \otimes \bar{m}_2 - m_2 \otimes \bar{b}_2\right) = -\tfrac{1}{2}\, b^-(m_2 \otimes \bar{m}_2)$$

obeys Leibniz rule on $\Delta = 0$, where B_1 and b^- anticommute.

B_2 obeys Jacobi for B_3 defined with $\mathrm{BV}^{\square}_{\infty}$ algebra of Yang-Mills; determines all data of gravity theory to *quartic* order.

E.g. cubic couplings:

$$\begin{aligned}\mathcal{L}^{(3)} = \tfrac{1}{8}\, e^{\mu\bar{\nu}}\Big(&\bar{\partial}^{\bar{\lambda}} e_{\mu\bar{\lambda}}\, \partial^{\rho} e_{\rho\bar{\nu}} + \partial^{\lambda} e_{\lambda\bar{\rho}}\, \bar{\partial}^{\bar{\rho}} e_{\mu\bar{\nu}} + 2\, \partial_{\mu} e_{\lambda\bar{\rho}}\, \bar{\partial}_{\bar{\nu}} e^{\lambda\bar{\rho}} \\ &- 2\, \partial_{\mu} e^{\lambda\bar{\rho}}\, \bar{\partial}_{\bar{\rho}} e_{\lambda\bar{\nu}} - 2\, \bar{\partial}_{\bar{\nu}} e^{\lambda\bar{\rho}}\, \partial_{\lambda} e_{\mu\bar{\rho}}\Big) \\ &+ \tfrac{1}{2}\, e^{\mu\bar{\nu}}\Big(f_{\mu} - \partial_{\mu} e\Big)\Big(\bar{f}_{\bar{\nu}} - \bar{\partial}_{\bar{\nu}} \bar{e}\Big)\end{aligned}$$

Only last line missed in ‘naive’ attempt.

Field-redefinition equivalent (but significant simplification!) to [Hull-Zwiebach (2009)].

DFT Gauge Transformations from Yang-Mills

Consider cubic vertex of Yang-Mills theory encoded in 2-bracket:

$$b_2(A,A)^a_\mu = f^a{}_{bc}(A^b \bullet A^c)_\mu$$

unambiguously defines 'kinematic' product

$$(v \bullet w)_\mu = v^\nu \partial_\nu w_\mu + (\partial_\mu v^\nu - \partial^\nu v_\mu) w_\nu + (\partial_\nu v^\nu) w_\mu - (v \leftrightarrow w)$$

Structure of generalized Lie derivative of DFT. In fact,

$$\delta^{(1)}_\lambda e_{\mu\bar{\mu}} = \tfrac{1}{4}(\lambda \bullet e_{\bar{\mu}})_\mu + \tfrac{1}{2}\lambda_\mu (\bar{f} - \bar{\partial}\,\bar{e})_{\bar{\mu}} \ ,$$

plus similar terms for $\bar{\lambda}_{\bar{\mu}}$.

Diffeomorphisms directly encoded in 3-vertex of Yang-Mills !!

Toward weakly constrained Double Field Theory

Subtlety of how to implement $\Delta = 0$ constraint.

- put $\square = \bar{\square}$ acting on fields, parameters, and their products,
  $\Rightarrow$ duality invariant way to identify coordinates x with $\bar{x}$
  $\Rightarrow$ *strongly constrained* DFT, reformulation of '$\mathcal{N} = 0$ SUGRA'

- *weakly constrained* DFT: $\square = \bar{\square}$ on fields but not products
  $\Rightarrow$ for tori T^d in principle obtainable from closed SFT by integrating out everything *except* KK and winding modes
  $\Rightarrow$ algebraic interpretation as *homotopy transfer*
  [Sen (2016), Arvanitakis, O. H., Hull, Lekeu (2020,2021)]

- explicit construction for toroidal dimensions: 1) total space $\mathcal{K} \otimes \bar{\mathcal{K}}$ of unconstrained functions of $(x, \bar{x})$ "BV_∞^Δ algebra";
  2) homotopy transfer to $\Delta = 0$ plus non-local shift of B_3
  $\Rightarrow$ genuine L_∞ algebra of weakly constrained fields
  [R. Bonezzi, C. Chiaffrino, F. Diaz-Jaramillo, work in progress]

Summary & Outlook I

- simplest double copy implementation at Lagrangian level:
  $\rightarrow$ gauge invariant DFT to quadratic order
  $\rightarrow$ Siegel gauge fixed DFT to cubic order

- Off-shell, local and gauge invariant double copy to *quartic order*:

  $\rightarrow$ Yang-Mills as homotopy Lie algebra or L_∞-algebra $\mathcal{K} \otimes \mathfrak{g}$ in terms of 'kinematic' C_∞-algebra $\mathcal{K}$

  $\rightarrow$ hidden 'Lie-type' algebra on $\mathcal{K}$: $\underline{\text{BV}^{\square}_{\infty}\text{ algebra}}$
  [Reiterer, 1912.03110 [math-ph]]

  $\rightarrow$ L_∞-algebra of DFT on $\mathcal{V} = \left[\mathcal{K} \otimes \bar{\mathcal{K}}\right]_{\text{level-matched}}$

- to all orders for strongly constrained DFT or $\mathcal{N} = 0$ SUGRA? would be complete first-principle derivation of double copy!

- classical solutions? $\rightarrow$ black holes!

Summary & Outlook II

- Can one include time (or more non-compact directions) in a weakly constrained DFT?
  For $\mu, \bar{\mu}, \ldots$ toroidal/spacelike quadratic Lagrangian

  $$\mathcal{L}^{(2)}_{\text{DFT}} = -(D_t\phi)^2 - D_t e^{\mu\bar{\nu}} D_t e_{\mu\bar{\nu}} + \mathcal{L}_{\text{spatial}}$$

  where

  $$D_t e_{\mu\bar{\nu}} := \partial_t e_{\mu\bar{\nu}} - \partial_\mu \bar{\mathcal{A}}_{\bar{\nu}} + \bar{\partial}_{\bar{\nu}} \mathcal{A}_\mu$$
  $$D_t\phi := \partial_t\phi + \partial_\mu \mathcal{A}^\mu + \bar{\partial}_{\bar{\mu}} \bar{\mathcal{A}}^{\bar{\mu}}$$

- weakly constrained DFT known to cubic order in time-dependent (FRW) backgrounds
  [O.H., Allison Pinto, 2207.14788]

- double copy or homotopy algebra techniques for *non-trivial backgrounds*?