NORMALIZATION OF THE VACUUM AND THE ULTRAVIOLET COMPLETION OF EINSTEIN GRAVITY

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Second-order-derivative plus fourth-order-derivative gravity is the ultraviolet completion of second-order-derivative quantum Einstein gravity. While it achieves renormalizability through states of negative Dirac norm, the unitarity violation that this would entail can be postponed to Planck energies. As we show in this paper the theory has a different problem, one that occurs at all energy scales, namely that the Dirac norm of the vacuum of the theory is not finite. To establish this we present a procedure for determining the norm of the vacuum in any quantum field theory. With the Dirac norm of the vacuum of the second-order-derivative plus fourth-order-derivative theory not being finite, the Feynman rules that are used to establish renormalizability are not valid, as is the assumption that the theory can be used as an effective theory at energies well below the Planck scale. This lack of finiteness is also manifested in the fact that the Minkowski path integral for the theory is divergent. Because the vacuum Dirac norm is not finite, the Hamiltonian of the theory is not Hermitian. However, it turns out to be PT symmetric. And when one continues the theory into the complex plane and uses the PT symmetry inner product, viz. the overlap of the left-eigenstate of the Hamiltonian with its right-eigenstate, one then finds that for the vacuum this norm is both finite and positive, the Feynman rules now are valid, the Minkowski path integral now is well behaved, and the theory now can serve as a low energy effective theory. Consequently, the theory can now be offered as a fully consistent, unitary and renormalizable theory of quantum gravity.

P. D. Mannheim, arXiv:2301.13029, arXiv:2303.10827

Outline

1. Is $\langle\Omega|\Omega\rangle$ finite? How could it not be? Specifying an action and canonical commutators does not fix a Hilbert space. We need to find a Hilbert space in which the Hamiltonian is self-adjoint. Setting $H_{ij} = H^*_{ji}$ only makes sense if H is self-adjoint when it acts on the (i, j) basis. Everything depends on boundary conditions.

2. We present a procedure to determine whether or not $\langle\Omega|\Omega\rangle$ is finite, and show that it is finite for a standard second-order derivative bosonic field theory. The procedure enables us to write the quantum field theory Hamiltonian as a first-quantized derivative operator.

3. We show that $\langle\Omega|\Omega\rangle$ is not finite for a fourth-order derivative bosonic field theory, and in the Hilbert space with the Dirac inner product the Hamiltonian is not self-adjoint.

4. We show that $\langle\Omega^L|\Omega^R\rangle = \langle\Omega^{CPT}|\Omega\rangle$ is finite for a fourth-order derivative bosonic field theory, and in this Hilbert space the Hamiltonian is self-adjoint, with there being no states of negative energy and no states of negative norm.

5. We show that $\langle\Omega|\Omega\rangle$ is finite for fermion theory.

6. We discuss our results from the perspective of path integrals and the Wick rotation to the Euclidean case, and show that in the fourth-order derivative bosonic field theory case the contribution of the Wick rotation contour circle at infinity is not only not zero, it is infinite.

7. We discuss our results from the perspective of the Dyson-Wick expansion.

8. We discuss the implications of our results for constructing a consistent theory of quantum gravity.

1 The hidden assumption of quantum field theory

Consider a free relativistic neutral scalar field with action

$$I_{\rm S}=\int d^4x\tfrac{1}{2}\left[\partial_\mu\phi\partial^\mu\phi-m^2\phi^2\right], \tag{1.1}$$

and wave equation, Hamiltonian, and equal time commutation relation of the form

$$\begin{aligned}[\partial_\mu\partial^\mu+m^2]\phi&=0,\\ H&=\int d^3x\tfrac{1}{2}[\dot{\phi}^2+\bar{\nabla}\phi\cdot\bar{\nabla}\phi+m^2\phi^2],\\ [\phi(\bar{x},t),\dot{\phi}(\bar{x}',t)]&=i\delta^3(\bar{x}-\bar{x}').\end{aligned} \tag{1.2}$$

With $\omega_k=+(\bar{k}^2+m^2)^{1/2}$ solutions to the wave equation obey

$$\phi(\bar{x},t)=\int\frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}}[a(\bar{k})e^{-i\omega_k t+i\bar{k}\cdot\bar{x}}+a^\dagger(\bar{k})e^{i\omega_k t-i\bar{k}\cdot\bar{x}}], \tag{1.3}$$

and with $[a(\bar{k}),a^\dagger(\bar{k}')]=\delta^3(\bar{k}-\bar{k}')$ the Hamiltonian is given by

$$H=\frac{1}{2}\int d^3k[\bar{k}^2+m^2]^{1/2}\left[a^\dagger(\bar{k})a(\bar{k})+a(\bar{k})a^\dagger(\bar{k})\right]. \tag{1.4}$$

Given (1.4) we can introduce a no-particle state $|\Omega\rangle$ that obeys $a(\bar{k})|\Omega\rangle=0$ for each $\bar{k}$, and can identify it as the ground state of H.

This procedure does not specify the value of $\langle\Omega|\Omega\rangle$.

For the theory the associated c-number propagator obeys

$$(\partial_t^2 - \bar{\nabla}^2 + m^2)D(x) = -\delta^4(x), \tag{1.5}$$

so that

$$D(x) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot x}}{(k^2 - m^2 + i\epsilon)}. \tag{1.6}$$

If we identify the propagator as a vacuum matrix element of q-number fields, viz.

$$D(x) = -i\langle\Omega|T[\phi(x)\phi(0)]|\Omega\rangle, \tag{1.7}$$

then use of the equal commutation relation gives

$$(\partial_t^2 - \bar{\nabla}^2 + m^2)(-i)\langle\Omega|T[\phi(x)\phi(0)]|\Omega\rangle = -\langle\Omega|\Omega\rangle\delta^4(x). \tag{1.8}$$

Comparing with (1.5) we see that we can only identify $D(x)$ as the matrix element $-i\langle\Omega|T[\phi(x)\phi(0)]|\Omega\rangle$ if the vacuum is normalized to one, viz. $\langle\Omega|\Omega\rangle = 1$.

Now if the normalization of the vacuum is finite we of course can always rescale it to one. However, that presupposes that the normalization of the vacuum is not infinite. We are not aware of any proof in the literature that the normalization of the vacuum is not infinite (either in this particular case or in general), and taking it to be finite is a **hidden assumption**.

So we shall present a procedure for determining whether the normalization of the vacuum state is finite or infinite. **The procedure is based on generalizing to quantum field theory what we know from quantum mechanics.**

2 The quantum-mechanical simple harmonic oscillator

For a simple harmonic oscillator with Hamiltonian $H = \frac{1}{2}[p^2 + q^2]$ and commutator $[q, p] = i$, there are two sets of bases, the wave function basis and the occupation number space basis.

The wave function basis is obtained by setting $p = -i\partial/\partial q$ in H and then solving the Schrödinger wave equation $H\psi(q) = E\psi(q)$. In this way we obtain a ground state with energy $E_0 = \frac{1}{2}$ and wave function $\psi_0(q) = e^{-q^2/2}$.

For occupation number space we set $q = (a + a^\dagger)/\sqrt{2}$ and $p = i(a^\dagger - a)/\sqrt{2}$. This yields $[a, a^\dagger] = 1$ and $H = a^\dagger a + 1/2$. We introduce a no-particle state $|\Omega\rangle$ that obeys $a|\Omega\rangle = 0$, with $|\Omega\rangle$ being the occupation number space ground state with energy $E_0 = \frac{1}{2}$. However, in and of itself this does not fix the norm $\langle\Omega|\Omega\rangle$ of the no-particle state or oblige it to be finite.

To fix the $\langle\Omega|\Omega\rangle$ norm we need to relate the ground states of the two bases. With $a = (q + ip)/\sqrt{2}$ we set

$$\langle q|a|\Omega\rangle = \frac{1}{\sqrt{2}}\left(q + \frac{\partial}{\partial q}\right)\langle q|\Omega\rangle = 0, \tag{2.1}$$

and find that $\langle q|\Omega\rangle = e^{-q^2/2}$. We thus identify $\psi_0(q) = \langle q|\Omega\rangle$. We now calculate the standard Dirac norm for vacuum, and obtain

$$\langle\Omega|\Omega\rangle = \int_{-\infty}^{\infty} dq\langle\Omega|q\rangle\langle q|\Omega\rangle = \int_{-\infty}^{\infty} dq\psi_0^*(q)\psi_0(q) = \int_{-\infty}^{\infty} dqe^{-q^2} = \sqrt{\pi}. \tag{2.2}$$

We thus establish that the Dirac norm of the no-particle state is finite. And on setting $\psi_0(q) = e^{-q^2/2}/\pi^{1/4}$ we normalize it to one.

That we are able to do this is because we know the form of the wave function $\psi_0(q)$.

While this procedure is both straightforward and familiar, it works because both the wave function basis approach and occupation number basis approach have something in common:

they are both based on an infinite number of degrees of freedom.

For the occupation number basis we can represent the creation and annihilation operators as infinite-dimensional matrices labeled by $|\Omega\rangle$, $a^\dagger|\Omega\rangle$, $a^{\dagger 2}|\Omega\rangle$ and so on.

For the wave function basis the coordinate q is a continuous variable that varies between $-\infty$ and ∞.

The two sets of bases are both infinite dimensional, one discrete and the other continuous.

The advantage of the continuous basis is that it enables to us to express the normalization of the vacuum state as an integral with an infinite range, an integral that is then either finite or infinite.

For field theory we already have an occupation number space basis for the Hamiltonian. So can we write it as a wave operator?

3 The quantum field theory oscillator

In the quantum field theory case we do not know the form of the wave function solutions to $H|\psi\rangle = E|\psi\rangle$, since we cannot realize the canonical commutator $[\phi(\bar{x},t),\dot{\phi}(\bar{x}',t)] = i\delta^3(\bar{x}-\bar{x}')$ as a differential relation. Specifically, we cannot satisfy it by setting $\dot{\phi}(\bar{x},t)$ equal to $-i\partial/\partial\phi(\bar{x},t)$ (though we could introduce a functional derivative $\dot{\phi}(\bar{x},t) = -i\delta/\delta\phi(\bar{x},t)$).

However, we can express the Hamiltonian in terms of creation and annihilation operators. So what we can then do is **reverse engineer** what we did in the quantum-mechanical case. For each $\bar{k}$ we thus introduce

$$a(\bar{k}) = \frac{1}{\sqrt{2}}[q(\bar{k}) + ip(\bar{k})], \quad a^\dagger(\bar{k}) = \frac{1}{\sqrt{2}}[q(\bar{k}) - ip(\bar{k})], \tag{3.1}$$

so that

$$[q(\bar{k}),p(\bar{k}')] = i\delta^3(\bar{k}-\bar{k}'), \quad H = \frac{1}{2}\int d^3k[\bar{k}^2+m^2]^{1/2}[p^2(\bar{k})+q^2(\bar{k})],$$
$$\phi(\bar{x},t) = \frac{1}{\sqrt{2}}\int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}}\left[[q(\bar{k})+ip(\bar{k})]e^{-i\omega_k t+i\bar{k}\cdot\bar{x}} + [q(\bar{k})-ip(\bar{k})]e^{i\omega_k t-i\bar{k}\cdot\bar{x}}\right]. \tag{3.2}$$

These $q(\bar{k})$ and $p(\bar{k})$ operators need not bear any relation to any physical position or momentum operators. Their only role here is to enable us to convert the discrete infinite-dimensional basis associated with each $a(\bar{k})$ and $a^\dagger(\bar{k})$ into a continuous one. Specifically, we can realize the $[q(\bar{k}),p(\bar{k}')]$ commutator by $p(\bar{k}') = -i\partial/\partial q(\bar{k}')$, with H then becoming a wave operator. In this way for each $\bar{k}$ we obtain a solution to the Schrödinger equation of the form $\psi(\bar{k}) = e^{-q^2(\bar{k})/2}/\pi^{1/4}$. We can define a no-particle vacuum that obeys $a(\bar{k})|\Omega\rangle$ for each $\bar{k}$. For each $\bar{k}$ we have

$$\langle q(\bar{k})|a(\bar{k})|\Omega\rangle = \frac{1}{\sqrt{2}}\left[q(\bar{k}) + \frac{\partial}{\partial q(\bar{k})}\right]\langle q(\bar{k})|\Omega\rangle = 0, \tag{3.3}$$

so that $\langle q(\bar{k})|\Omega\rangle = e^{-q^2(\bar{k})/2}/\pi^{1/4}$, and thus

$$\langle\Omega|\Omega\rangle = \Pi_{\bar{k}}\int dq(\bar{k})\langle\Omega|q(\bar{k})\rangle\langle q(\bar{k})|\Omega\rangle = \Pi_{\bar{k}}\int dq(\bar{k})\frac{e^{-q^2(\bar{k})}}{\pi^{1/2}} = \Pi_{\bar{k}}1 = 1. \tag{3.4}$$

Thus the vacuum for the full H obeys $\langle\Omega|\Omega\rangle = 1$, to thus have a finite normalization. In this way we establish that the vacuum state of the free relativistic scalar field is normalizable.

Once we are able to show that the vacuum state of the free theory is normalizable, this will remain true in the presence of interactions if the interacting theory is renormalizable. To see this we note that in developing Wick's contraction theorem in quantum field theory one needs to put the time-ordered product of Heisenberg fields $\langle\Omega|T[\phi(x_1)...\phi(x_n)]|\Omega\rangle$ into a form that can be developed perturbatively. To this end one introduces a set of in-fields $\phi_{in}(x)$ that satisfy free field equations with Hamiltonian H_{in}. And one also introduces an evolution operator $U(t)$ that evolves the interaction Hamiltonian $H_I(t)$ and fields according to

$$i\frac{\partial U(t)}{\partial t}=H_I(t)U(t),\qquad \phi(\bar{x},t)=U^{-1}(t)\phi_{in}(\bar{x},t)U(t). \tag{3.5}$$

Using these relations we obtain

$$\langle\Omega|T[\phi(x_1)...\phi(x_n)]|\Omega\rangle=\langle\Omega|T\left[\phi_{in}(x_1)...\phi_{in}(x_n)\exp\left(-i\int_{-t}^{t}dt_1H_I(t_1)\right)\right]|\Omega\rangle\langle\Omega|T\left[\exp\left(i\int_{-t}^{t}dt_1H_I(t_1)\right)\right]|\Omega\rangle. \tag{3.6}$$

After inverting the last term we obtain the standard form for the perturbative Wick contraction procedure, viz.

$$\langle\Omega|T[\phi(x_1)...\phi(x_n)]|\Omega\rangle=\frac{\langle\Omega|T\left[\phi_{in}(x_1)...\phi_{in}(x_n)\exp\left(-i\int_{-t}^{t}dt_1H_I(t_1)\right)\right]|\Omega\rangle}{\langle\Omega|T\left[\exp\left(-i\int_{-t}^{t}dt_1H_I(t_1)\right)\right]|\Omega\rangle}. \tag{3.7}$$

If one starts with (3.7) it would appear that the normalization of the vacuum state is actually irrelevant since it would drop out of the ratio. And so it would not appear to matter if it did happen to be infinite. However, this is not the case since we could only go from (3.6) to (3.7) if $\langle\Omega|T\left[\exp\left(i\int_{-t}^{t}dt_1H_I(t_1)\right)\right]|\Omega\rangle$ is finite. And it would not be if the vacuum state is not normalizable. If we expand $\langle\Omega|T\left[\exp\left(i\int_{-t}^{t}dt_1H_I(t_1)\right)\right]|\Omega\rangle$ out as a power series in H_I the first term is $\langle\Omega|\Omega\rangle$ as calculated in a free theory. Thus, for finiteness we first need this term to be finite and then need the power series expansion in H_I to be renormalizable in order for the interacting $\langle\Omega|T\left[\exp\left(i\int_{-t}^{t}dt_1H_I(t_1)\right)\right]|\Omega\rangle$ to be finite too. However, for a nonnormalizable vacuum the standard Wick expansion and Feynman rules that are obtained from (3.7) are not valid. Since this concern is of relevance to radiative corrections to Einstein gravity we return to this point below.

As well as providing a procedure for determining whether or not $\langle\Omega|\Omega\rangle$ is finite, since the procedure enables is to express the free second-order-derivative Hamiltonian H as an ordinary derivative operator, it does so for interactions as well. Specifically, from (3.2) we can write $\phi(\bar{x},t)$ as a derivative operator, viz.

$$\phi(\bar{x},t)=\frac{1}{\sqrt{2}}\int\frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}}\left[\left[q(\bar{k})+\frac{\partial}{\partial q(\bar{k})}\right]e^{-i\omega_k t+i\bar{k}\cdot\bar{x}}+\left[q(\bar{k})-\frac{\partial}{\partial q(\bar{k})}\right]e^{i\omega_k t-i\bar{k}\cdot\bar{x}}\right]. \tag{3.8}$$

Thus the insertion of (3.8) into an interaction Hamiltonian of the form $H_I=\lambda\int d^3x\phi^4(\bar{x},t)$ enables us to write H_I, and thus $H+H_I$, as a derivative operator. While this procedure enables us to in principle set up the Schrödinger problem for $H+H_I$ as a wave mechanics problem, it is still quite a formidable one, just as interacting field theories always have been.

Fermions

For fermions we have to deal with anticommutators such as

$$bb^\dagger+b^\dagger b=1. \tag{3.9}$$

Also, because of the Pauli principle we have

$$b^2=0,\qquad b^{\dagger 2}=0. \tag{3.10}$$

We can represent (3.9) and (3.10) by matrices of the form

$$b=\begin{pmatrix}0&0\\1&0\end{pmatrix},\qquad b^\dagger=\begin{pmatrix}0&1\\0&0\end{pmatrix}. \tag{3.11}$$

Thus, unlike the infinite-dimensional matrix representation of the bosonic a and $a^\dagger$ that obey $aa^\dagger-a^\dagger a=1$, the fermionic b and $b^\dagger$ matrices are finite dimensional. Thus with a finite number of degrees of freedom, the fermion vacuum that obeys $b|\Omega\rangle=0$ has a finite $\langle\Omega|\Omega\rangle$ norm.

4 Higher-derivative quantum field theories

The action and equation of motion are of the form

$$
\begin{aligned}
I_S &= \frac{1}{2}\int d^4x\Big[\partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi-(M_1^2+M_2^2)\partial_\mu\phi\partial^\mu\phi+M_1^2M_2^2\phi^2\Big],\\
&(\partial_t^2-\bar{\nabla}^2+M_1^2)(\partial_t^2-\bar{\nabla}^2+M_2^2)\phi(x)=0.
\end{aligned}
\tag{4.1}
$$

The associated propagator obeys the ghost-like

$$
\begin{aligned}
&(\partial_t^2-\bar{\nabla}^2+M_1^2)(\partial_t^2-\bar{\nabla}^2+M_2^2)D(x)=-\delta^4(x),\\
&D(x)=-\int\frac{d^4k}{(2\pi)^4}\frac{e^{-ik\cdot x}}{(k^2-M_1^2)(k^2-M_2^2)}=-\int\frac{d^4k}{(2\pi)^4}\frac{e^{-ik\cdot x}}{(M_1^2-M_2^2)}\left[\frac{1}{(k^2-M_1^2)}-\frac{1}{(k^2-M_2^2)}\right].
\end{aligned}
\tag{4.2}
$$

The energy-momentum tensor $T_{\mu\nu}$, the canonical momenta π^μ and $\pi^{\mu\lambda}$, and the equal-time commutators appropriate to the higher-derivative theory are given by (Bender and Mannheim 2008)

$$
\begin{aligned}
T_{\mu\nu}&=\pi_\mu\phi_{,\nu}+\pi_\mu{}^\lambda\phi_{,\nu,\lambda}-\eta_{\mu\nu}\mathcal{L},\\
\pi^\mu&=\frac{\partial\mathcal{L}}{\partial\phi_{,\mu}}-\partial_\lambda\left(\frac{\partial\mathcal{L}}{\partial\phi_{,\mu,\lambda}}\right)=-\partial_\lambda\partial^\mu\partial^\lambda\phi-(M_1^2+M_2^2)\partial^\mu\phi,\\
\pi^{\mu\lambda}&=\frac{\partial\mathcal{L}}{\partial\phi_{,\mu,\lambda}}=\partial^\mu\partial^\lambda\phi,\\
T_{00}&=\tfrac{1}{2}\pi_{00}^2+\pi_0\dot{\phi}+\tfrac{1}{2}(M_1^2+M_2^2)\dot{\phi}^2-\tfrac{1}{2}M_1^2M_2^2\phi^2-\tfrac{1}{2}\pi_{ij}\pi^{ij}+\tfrac{1}{2}(M_1^2+M_2^2)\phi_{,i}\phi^{,i}\\
&=\frac{1}{2}\ddot{\phi}^2-\tfrac{1}{2}(M_1^2+M_2^2)\dot{\phi}^2-\dddot{\phi}\dot{\phi}-[\partial_i\partial^i\dot{\phi}]\dot{\phi}-\tfrac{1}{2}M_1^2M_2^2\phi^2-\tfrac{1}{2}\partial_i\partial_j\phi\partial^i\partial^j\phi+\tfrac{1}{2}(M_1^2+M_2^2)\partial_i\phi\partial^i\phi,\\
&[\phi(\bar{0},t),\dot{\phi}(\bar{x},t)]=0,\qquad[\phi(\bar{0},t),\ddot{\phi}(\bar{x},t)]=0,\qquad[\phi(\bar{0},t),\dddot{\phi}(\bar{x},t])=-i\delta^3(x).
\end{aligned}
\tag{4.3}
$$

With the use of these commutation relations we find that

$$
D(x)=i\langle\Omega|T[\phi(x)\phi(0)]|\Omega\rangle
\tag{4.4}
$$

indeed satisfies the first equation given in (4.2), **provided that is that $\langle\Omega|\Omega\rangle=1$. And if $\langle\Omega|\Omega\rangle=\infty$, then Wick's theorem and the associated Feynman rules are not valid.**

To check whether $\langle\Omega|\Omega\rangle$ actually is finite, we need to express the scalar field Hamiltonian $H_S = \int d^3x T_{00}$ in terms of creation and annihilation operators and then construct an equivalent wave mechanics. Given that the solutions to to the wave equation are plane waves, we set

$$\phi(\bar{x},t) = \int \frac{d^3k}{(2\pi)^{3/2}} \left[a_1(\bar{k})e^{-i\omega_1 t+i\bar{k}\cdot\bar{x}} + a_1^\dagger(\bar{k})e^{i\omega_1 t-i\bar{k}\cdot\bar{x}} + a_2(\bar{k})e^{-i\omega_2 t+i\bar{k}\cdot\bar{x}} + a_2^\dagger(\bar{k})e^{i\omega_2 t-i\bar{k}\cdot\bar{x}}\right]. \tag{4.5}$$

where $\omega_1 = +(\bar{k}^2 + M_1^2)^{1/2}$, $\omega_2 = +(\bar{k}^2 + M_2^2)^{1/2}$. Given the commutators in (4.3) we obtain

$$\begin{aligned}
&[a_1(\bar{k}), a_1^\dagger(\bar{k}')] = [2(M_1^2 - M_2^2)(\bar{k}^2 + M_1^2)^{1/2}]^{-1}\delta^3(\bar{k}-\bar{k}'),\\
&[a_2(\bar{k}), a_2^\dagger(\bar{k}')] = -[2(M_1^2 - M_2^2)(\bar{k}^2 + M_2^2)^{1/2}]^{-1}\delta^3(\bar{k}-\bar{k}'),\\
&[a_1(\bar{k}), a_2(\bar{k}')] = 0, \quad [a_1(\bar{k}), a_2^\dagger(\bar{k}')] = 0, \quad [a_1^\dagger(\bar{k}), a_2(\bar{k}')] = 0, \quad [a_1^\dagger(\bar{k}), a_2^\dagger(\bar{k}')] = 0,
\end{aligned} \tag{4.6}$$

with the Hamiltonian then taking the form

$$\begin{aligned}
H_S &= \frac{1}{2}\int d^3k \bigg[2(M_1^2 - M_2^2)(\bar{k}^2 + M_1^2)\left[a_1^\dagger(\bar{k})a_1(\bar{k}) + a_1(\bar{k})a_1^\dagger(\bar{k})\right]\\
&- 2(M_1^2 - M_2^2)(\bar{k}^2 + M_2^2)\left[a_2^\dagger(\bar{k})a_2(\bar{k}) + a_2(\bar{k})a_2^\dagger(\bar{k})\right]\bigg]\\
&= \int d^3k \bigg[2(M_1^2 - M_2^2)(\bar{k}^2 + M_1^2)a_1^\dagger(\bar{k})a_1(\bar{k}) - 2(M_1^2 - M_2^2)(\bar{k}^2 + M_2^2)a_2^\dagger(\bar{k})a_2(\bar{k})\\
&+ \frac{1}{2}(\bar{k}^2 + M_1^2)^{1/2}\delta^3(0) + \frac{1}{2}(\bar{k}^2 + M_2^2)^{1/2}\delta^3(0)\bigg],
\end{aligned} \tag{4.7}$$

where $(2\pi)^3\delta^3(0)$ is a quantization box volume V. We note that with $M_1^2 - M_2^2 > 0$ for definitiveness, we see **negative signs** in both H_S and the $[a_2(\bar{k}), a_2^\dagger(\bar{k}')]$ commutator, while noting that despite this the zero-point energy is **positive**. We shall see below that the negative sign concerns will be resolved once we settle the issue of the normalization of the vacuum. To do that we now descend to the quantum-mechanical limit of the theory, the Pais-Uhlenbeck oscillator model.

5 Higher-derivative quantum mechanics

In order to study the Pauli-Villars regulator, in 1950 Pais and Uhlenbeck (PU) introduced a fourth-order quantum-mechanical oscillator model with action and equation of motion

$$I_{\rm PU} = \frac{1}{2}\int dt\left[\ddot{z}^2 - \left(\omega_1^2+\omega_2^2\right)\dot{z}^2 + \omega_1^2\omega_2^2 z^2\right], \qquad \ddddot{z} + (\omega_1^2+\omega_2^2)\ddot{z} + \omega_1\omega_2 z^2 = 0, \tag{5.1}$$

where for definitiveness in the following we take $\omega_1 > \omega_2$. This action is just the scalar field theory action

$$I_S = \frac{1}{2}\int d^4x\left[\partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi - (M_1^2+M_2^2)\partial_\mu\phi\partial^\mu\phi + M_1^2M_2^2\phi^2\right] \tag{5.2}$$

with the spatial dependence frozen out.

As constructed this action possesses three variables z, $\dot{z}$ and $\ddot{z}$. This is too many for one oscillator but not enough for two. The system is thus a constrained system. And so we introduce a new variable $x=\dot{z}$ and its conjugate p_x. And using the method of Dirac constraints obtain the time-independent Hamiltonian (Mannheim and Davidson 2000, 2005)

$$H_{\rm PU} = \frac{p_x^2(t)}{2} + p_z(t)x(t) + \frac{1}{2}\left(\omega_1^2+\omega_2^2\right)x^2(t) - \frac{1}{2}\omega_1^2\omega_2^2 z^2(t), \tag{5.3}$$

with two sets of canonical equal-time commutators of the form

$$[z(t), p_z(t)] = i, \qquad [x(t), p_x(t)] = i. \tag{5.4}$$

On setting $p_z = -i\partial_z$, $p_x = -i\partial_x$ the Schrödinger problem for $H_{\rm PU}$ can be solved analytically, with the state with energy $(\omega_1+\omega_2)/2$ having a wave function that is of the form (Mannheim 2007)

$$\psi_0(z,x) = \exp[\tfrac{1}{2}(\omega_1+\omega_2)\omega_1\omega_2 z^2 + i\omega_1\omega_2 zx - \tfrac{1}{2}(\omega_1+\omega_2)x^2]. \tag{5.5}$$

While this wave function is well behaved at large x, it diverges at large z, and consequently as a wave function it is not normalizable. The state with energy $(\omega_1+\omega_2)/2$ is the lowest energy state of an infinite tower of positive energy modes.

However, the state with energy $(\omega_1-\omega_2)/2$ has a wave function that is of the form (Bender and Mannheim 2008)

$$\psi_0'(z,x) = \exp[-\tfrac{1}{2}(\omega_1-\omega_2)\omega_1\omega_2 z^2 - i\omega_1\omega_2 zx - \tfrac{1}{2}(\omega_1-\omega_2)x^2]. \tag{5.6}$$

This wave function is well behaved at large x and at large z, and consequently as a wave function it is normalizable. The state with energy $(\omega_1-\omega_2)/2$ lies in the middle of an infinite tower of positive energy modes and an infinite tower of negative energy modes - the Ostrogradski instability.

To relate the $\psi_0(z,x)$ wave function to the no-particle vacuum $|\Omega\rangle$ we second quantize the theory. And with the wave equation given in (5.1), and with $\dot{z}=i[H_{\rm PU},z]=x$, $\dot{x}=p_x$, $\dot{p}_x=-p_z-(\omega_1^2+\omega_2^2)x$, $\dot{p}_z=\omega_1^2\omega_2^2 z$, we obtain

$$
\begin{aligned}
z(t) &= a_1e^{-i\omega_1 t}+a_1^{\dagger}e^{i\omega_1 t}+a_2e^{-i\omega_2 t}+a_2^{\dagger}e^{i\omega_2 t},\\
p_z(t) &= i\omega_1\omega_2^2[a_1e^{-i\omega_1 t}-a_1^{\dagger}e^{i\omega_1 t}]+i\omega_1^2\omega_2[a_2e^{-i\omega_2 t}-a_2^{\dagger}e^{i\omega_2 t}],\\
x(t) &= -i\omega_1[a_1e^{-i\omega_1 t}-a_1^{\dagger}e^{i\omega_1 t}]-i\omega_2[a_2e^{-i\omega_2 t}-a_2^{\dagger}e^{i\omega_2 t}],\\
p_x(t) &= -\omega_1^2[a_1e^{-i\omega_1 t}+a_1^{\dagger}e^{i\omega_1 t}]-\omega_2^2[a_2e^{-i\omega_2 t}+a_2^{\dagger}e^{i\omega_2 t}],\\
a_1e^{-i\omega_1 t} &= \frac{1}{2(\omega_1^2-\omega_2^2)}\left[-\omega_2^2 z(t)-p_x(t)+i\omega_1 x(t)+i\frac{p_z(t)}{\omega_1}\right],\\
a_1^{\dagger}e^{i\omega_1 t} &= \frac{1}{2(\omega_1^2-\omega_2^2)}\left[-\omega_2^2 z(t)-p_x(t)-i\omega_1 x(t)-i\frac{p_z(t)}{\omega_1}\right],\\
a_2e^{-i\omega_2 t} &= \frac{1}{2(\omega_1^2-\omega_2^2)}\left[\omega_1^2 z(t)+p_x(t)-i\omega_2 x(t)-i\frac{p_z(t)}{\omega_2}\right],\\
a_2^{\dagger}e^{i\omega_2 t} &= \frac{1}{2(\omega_1^2-\omega_2^2)}\left[\omega_1^2 z(t)+p_x(t)+i\omega_2 x(t)+i\frac{p_z(t)}{\omega_2}\right],
\end{aligned}
\tag{5.7}
$$

and a Hamiltonian and commutator algebra of the form (Mannheim and Davidson 2000, 2005)

$$
H_{\rm PU}=2(\omega_1^2-\omega_2^2)(\omega_1^2a_1^{\dagger}a_1-\omega_2^2a_2^{\dagger}a_2)+\tfrac{1}{2}(\omega_1+\omega_2),
\tag{5.8}
$$

$$
[a_1,a_1^{\dagger}]=\frac{1}{2\omega_1(\omega_1^2-\omega_2^2)},\qquad [a_2,a_2^{\dagger}]=-\frac{1}{2\omega_2(\omega_1^2-\omega_2^2)}.
\tag{5.9}
$$

We note the similarity to (4.7) and (4.6).

The no-particle state $|\Omega\rangle$ that both a_1 and a_2 annihilate is the state with energy $(\omega_1+\omega_2)/2$. And with its energy being $(\omega_1+\omega_2)/2$, we can associate it with $\psi_0(z,x)e^{-i(\omega_1+\omega_2)t/2}$, with the normalization of $|\Omega\rangle$ then being given by

$$
\langle\Omega|\Omega\rangle=\int_{-\infty}^{\infty}dz\int_{-\infty}^{\infty}dx\langle\Omega|z,x\rangle\langle z,x|\Omega\rangle=\int_{-\infty}^{\infty}dz\int_{-\infty}^{\infty}dx\psi_0^*(z,x)\psi_0(z,x)=\infty.
\tag{5.10}
$$

With $\psi_0(z,x)$ diverging at large z, this normalization integral is infinite. Thus we see that through our knowledge of the form of the ground state wave function $\psi_0(z,x)$ we are able to establish that $\langle\Omega|\Omega\rangle$ is **infinite**. By the same token, for the state $|\Omega'\rangle$ that both a_1 and $a_2^{\dagger}$ annihilate, the $\langle\Omega'|\Omega'\rangle$ norm is **finite**.

6 The nonnormalizable vacuum of higher-derivative field theories

On generalizing to each $\bar{k}$ and setting $\omega_1(\bar{k})=+(\bar{k}^2+M_1^2)^{1/2}$, $\omega_2(\bar{k})=+(\bar{k}^2+M_2^2)^{1/2}$, we obtain

$$
\begin{aligned}
a_1(\bar{k})e^{-i\omega_1(\bar{k})t} &= \frac{1}{2(M_1^2-M_2^2)}\left[-\omega_2^2(\bar{k})z(\bar{k},t)-p_x(\bar{k},t)+i\omega_1(\bar{k})x(\bar{k},t)+i\frac{p_z(\bar{k},t)}{\omega_1(\bar{k})}\right],\\
a_1^\dagger(\bar{k})e^{i\omega_1(\bar{k})t} &= \frac{1}{2(M_1^2-M_2^2)}\left[-\omega_2^2(\bar{k})z(\bar{k},t)-p_x(\bar{k},t)-i\omega_1(\bar{k})x(\bar{k},t)-i\frac{p_z(\bar{k},t)}{\omega_1(\bar{k})}\right],\\
a_2(\bar{k})e^{-i\omega_2(\bar{k})t} &= \frac{1}{2(M_1^2-M_2^2)}\left[\omega_1^2(\bar{k})z(\bar{k},t)+p_x(\bar{k},t)-i\omega_2(\bar{k})x(\bar{k},t)-i\frac{p_z(\bar{k},t)}{\omega_2(\bar{k})}\right],\\
a_2^\dagger(\bar{k})e^{i\omega_2(\bar{k})t} &= \frac{1}{2(M_1^2-M_2^2)}\left[\omega_1^2(\bar{k})z(\bar{k},t)+p_x(\bar{k},t)+i\omega_2(\bar{k})x(\bar{k},t)+i\frac{p_z(\bar{k},t)}{\omega_2(\bar{k})}\right].
\end{aligned}
\tag{6.1}
$$

Inverting (6.1) gives

$$
\begin{aligned}
z(\bar{k},t) &= a_1(\bar{k})e^{-i\omega_1(\bar{k})t}+a_1^\dagger(\bar{k})e^{i\omega_1(\bar{k})t}+a_2(\bar{k})e^{-i\omega_2(\bar{k})t}+a_2^\dagger(\bar{k})^{i\omega_2(\bar{k})t},\\
p_z(\bar{k},t) &= i\omega_1(\bar{k})\omega_2^2(\bar{k})[a_1(\bar{k})e^{-i\omega_1(\bar{k})t}-a_1^\dagger(\bar{k})e^{i\omega_1(\bar{k})t}]+i\omega_1^2(\bar{k})\omega_2(\bar{k})[a_2(\bar{k})e^{-i\omega_2(\bar{k})t}-a_2^\dagger(\bar{k})e^{i\omega_2(\bar{k})t}],\\
x(\bar{k},t) &= -i\omega_1(\bar{k})[a_1(\bar{k})e^{-i\omega_1(\bar{k})t}-a_1^\dagger(\bar{k})e^{i\omega_1(\bar{k})t}]-i\omega_2(\bar{k})[a_2(\bar{k})e^{-i\omega_2(\bar{k})t}-a_2^\dagger(\bar{k})^{i\omega_2(\bar{k})t}],\\
p_x(\bar{k},t) &= -\omega_1^2(\bar{k})[a_1(\bar{k})e^{-i\omega_1(\bar{k})t}+a_1^\dagger(\bar{k})e^{i\omega_1(\bar{k})t}]-\omega_2^2(\bar{k})[a_2(\bar{k})e^{-i\omega_2(\bar{k})t}+a_2^\dagger(\bar{k})^{i\omega_2(\bar{k})t}].
\end{aligned}
\tag{6.2}
$$

From (6.2) and the commutation relations given in (4.6) it follows that

$$
\begin{aligned}
&[z(\bar{k},t),p_z(\bar{k}',t)]=\delta^3(\bar{k}-\bar{k}'),\qquad [x(\bar{k},t),p_x(\bar{k}',t)]=\delta^3(\bar{k}-\bar{k}'),\\
&[z(\bar{k},t),x(\bar{k}',t)]=0,\quad [z(\bar{k},t),p_x(\bar{k}',t)]=0,\quad [p_z(\bar{k},t),x(\bar{k}',t)]=0,\quad [p_z(\bar{k},t),p_x(\bar{k}',t)]=0.
\end{aligned}
\tag{6.3}
$$

Insertion of (6.1) into the Hamiltonian given in (4.7) then yields an equivalent, time-independent Hamiltonian

$$
H_S = \int d^3k\left[\frac{p_x^2(\bar{k},t)}{2}+p_z(\bar{k},t)x(\bar{k},t)+\frac{1}{2}\left[\omega_1^2(\bar{k})+\omega_2^2(\bar{k})\right]x^2(\bar{k},t)-\frac{1}{2}\omega_1^2(\bar{k})\omega_2^2(\bar{k})z^2(\bar{k},t)\right].
\tag{6.4}
$$

For each momentum state we recognize the quantum field theory Hamiltonian H_S given in (6.4) as being of precisely the form of the quantum-mechanical $H_{\rm PU}$ Hamiltonian that is given in (5.3). Thus the frozen out spatial dependence reemerges as a momentum dependence.

We can now proceed as in the second-order case and represent the commutators by

$$\left[z(\bar{k},t),-i\frac{\partial}{\partial z(\bar{k}',t)}\right]=\delta^3(\bar{k}-\bar{k}'),\qquad \left[x(\bar{k},t),-i\frac{\partial}{\partial x(\bar{k}',t)}\right]=\delta^3(\bar{k}-\bar{k}'). \tag{6.5}$$

With the vacuum obeying $a_1(\bar{k})|\Omega\rangle=0$, $a_2(\bar{k})|\Omega\rangle=0$ for each $\bar{k}$, from (6.1) we obtain

$$\langle z(\bar{k}),x(\bar{k})|a_1(\bar{k})|\Omega\rangle=\frac{1}{2(M_1^2-M_2^2)}\left[-\omega_2^2(\bar{k})z(\bar{k})+i\frac{\partial}{\partial x(\bar{k})}+i\omega_1(\bar{k})x(\bar{k})+\frac{1}{\omega_1(\bar{k})}\frac{\partial}{\partial z(\bar{k})}\right]\langle z(\bar{k}),x(\bar{k})|\Omega\rangle=0,$$
$$\langle z(\bar{k}),x(\bar{k})|a_2(\bar{k})|\Omega\rangle=\frac{1}{2(M_1^2-M_2^2)}\left[\omega_1^2(\bar{k})z(\bar{k})-i\frac{\partial}{\partial x(\bar{k})}-i\omega_2(\bar{k})x(\bar{k})-\frac{1}{\omega_2(\bar{k})}\frac{\partial}{\partial z(\bar{k})}\right]\langle z(\bar{k}),x(\bar{k})|\Omega\rangle=0, \tag{6.6}$$

for each $\bar{k}$. From (6.6) it follows that for each $\bar{k}$ we can identify each $\langle z(\bar{k}),x(\bar{k})|\Omega\rangle$ with the PU oscillator ground state wave function $\psi_0(z(\bar{k}),x(\bar{k}))$, which, analogously to (5.5), is given by

$$\psi_0(z(\bar{k}),x(\bar{k}))=\exp[\tfrac{1}{2}[\omega_1(\bar{k})+\omega_2(\bar{k})]\omega_1(\bar{k})\omega_2(\bar{k})z^2(\bar{k})+i\omega_1(\bar{k})\omega_2(\bar{k})z(\bar{k})x(\bar{k})-\tfrac{1}{2}[\omega_1(\bar{k})+\omega_2(\bar{k})]x^2(\bar{k})]. \tag{6.7}$$

Consequently, the normalization of the vacuum is given by

$$\begin{aligned}\langle\Omega|\Omega\rangle&=\Pi_{\bar{k}}\int_{-\infty}^{\infty}dz(\bar{k})\int_{-\infty}^{\infty}dx(\bar{k})\langle\Omega|z(\bar{k}),x(\bar{k})\rangle\langle z(\bar{k}),x(\bar{k})|\Omega\rangle\\&=\Pi_{\bar{k}}\int_{-\infty}^{\infty}dz(\bar{k})\int_{-\infty}^{\infty}dx(\bar{k})\psi_0^*(z(\bar{k}),x(\bar{k}))\psi_0(z(\bar{k}),x(\bar{k})).\end{aligned} \tag{6.8}$$

With each $\psi_0(z(\bar{k}),x(\bar{k}))$ diverging at large $z(\bar{k})$, **we thus establish that the Dirac norm of the field theory $|\Omega\rangle$ vacuum is infinite**. Thus whatever is the normalization of the vacuum in the associated wave-mechanical limit translates into the same normalization in the quantum field theory.

The Different Realizations of the Higher-Derivative Theory

$$(\partial_t^2 - \bar{\nabla}^2 + M_1^2)(\partial_t^2 - \bar{\nabla}^2 + M_2^2)\phi(x) = 0,$$
$$\phi(\bar{x},t) = \frac{1}{(2\pi)^{3/2}}\int d^3k \left[a_1(\bar{k})e^{-i\omega_1 t + i\bar{k}\cdot\bar{x}} + a_1^\dagger(\bar{k})e^{i\omega_1 t - i\bar{k}\cdot\bar{x}} + a_2(\bar{k})e^{-i\omega_2 t + i\bar{k}\cdot\bar{x}} + a_2^\dagger(\bar{k})e^{i\omega_2 t - i\bar{k}\cdot\bar{x}}\right]. \quad (6.9)$$

Two Different Creation and Annihilation Realizations

$a_2(\bar{k})|\Omega\rangle = 0$: Energies **bounded** from below but $\langle\Omega|a_2(\bar{k})a_2^\dagger(\bar{k})|\Omega\rangle$ is both **infinite** and **negative**.

$a_2^\dagger(\bar{k})|\Omega'\rangle = 0$: Energies **unbounded** from below but $\langle\Omega'|a_2^\dagger(\bar{k})a_2(\bar{k})|\Omega'\rangle$ is both **finite** and **positive**.

These two realizations occur in **different** Hilbert spaces, so in any given Hilbert space only one problem, negative energies or negative norms but not both. With negative energies being unacceptable (the Ostrogradski instability), we have to work in Hilbert space with bounded from below energy spectrum and within it we have two problems to solve.

Two Different Feynman Contour $i\epsilon$ Realizations

Annihilation operators go with the positive frequency modes and the creation operators go with the negative frequency modes, viz.

$$D(k) = \frac{1}{k^2 - M_1^2 + i\epsilon} - \frac{1}{k^2 - M_2^2 + i\epsilon} \quad (6.10)$$

so that positive frequencies propagate forward in time and negative frequencies propagate backward in time, no Ostrogradski instability. No negative energies but negative residues.

There is an alternate $i\epsilon$ prescription of the form

$$D'(k) = \frac{1}{k^2 - M_1^2 + i\epsilon} - \frac{1}{k^2 - M_2^2 - i\epsilon}, \quad (6.11)$$

so that while the positive frequencies associated with the M_1 sector propagate forward in time and negative frequencies propagate backward in time, for the M_2 sector **positive frequencies propagate backward in time** and **negative frequencies propagate forward in time**. The energy spectrum is now unbounded from below, but no negative residues. And now $a_2^\dagger(\bar{k})$ annihilates the no-particle state rather then $a_2(\bar{k})$. This realization is not physically acceptable.

The Takeaway

Write the quantum field theory Hamiltonian in terms of creation and annihilation operators. Change the basis to analog position and momentum operators. Realize the momentum operators as differential operators and set up an analog Schrödinger problem. Then check to see whether the wave functions are normalizable.

And if the wave functions are not normalizable?

Then we cannot integrate by parts and throw away surface terms. Then the **Hamiltonian is not Hermitian or self-adjoint**. So we cannot use the Dirac inner product, since

$$\langle\Omega(t)|\Omega(t)\rangle = \langle\Omega(t=0)|e^{iH^\dagger t}e^{-iHt}|\Omega(t=0)\rangle \neq \langle\Omega(t=0)|\Omega(t=0)\rangle, \tag{6.12}$$

to thus not be time independent, and thus not acceptable.

Can we find an inner product that is time independent?

As shown in Mannheim 2018 the most general inner product that one could use is the overlap of the left and right eigenstates of the Hamiltonian (the largest set of eigenstates that a Hamiltonian can have), i.e., states that satisfy

$$-i\frac{\partial}{\partial t}\langle L(t)| = \langle L(t)|H, \qquad i\frac{\partial}{\partial t}|R(t)\rangle = H|R(t)\rangle, \tag{6.13}$$

$$\langle L(t)|R(t)\rangle = \langle L(t=0)|e^{iHt}e^{-iHt}|R(t=0)\rangle = \langle L(t=0)|R(t=0)\rangle, \tag{6.14}$$

to thus be time independent. This is also the same as

$$\langle R^{CPT}(t)|R(t)\rangle = \langle L(t)|R(t)\rangle. \tag{6.15}$$

Fine, but is this inner product finite and is it positive?

7 How to obtain a normalizable vacuum

In analyzing the second-order plus fourth-order scalar field theory we note that with a conventional Hermitian field $\phi(x)$, and thus with $a_1^\dagger(\bar{k})$ and $a_2^\dagger(\bar{k})$ being the Hermitian conjugates of $a_1(\bar{k})$ and $a_2(\bar{k})$, the $a_2^\dagger(\bar{k})a_2(\bar{k})$ product would be positive definite and the energy spectrum of H_S as given in (4.7) would initially be unbounded from below, this being the familiar Ostrogradski instability of higher-derivative theories with Hermitian fields.

However, from (4.6) we see that $\langle\Omega|a_2(\bar{k})a_2^\dagger(\bar{k})|\Omega\rangle$ would be negative. This would imply the presence of ghost states of negative norm, with it then not being the case that a product such as $a_2(\bar{k})a_2^\dagger(\bar{k})$ could be positive definite. If one accepts this then matrix elements of the $-2(M_1^2 - M_2^2)(\bar{k}^2 + M_2^2)a_2^\dagger(\bar{k})a_2(\bar{k})$ term in H_S would be compensated for by the ghost signature, and the energy spectrum of H_S would then be bounded from below.

While this takes care of the unboundedness of the energy spectrum, it does so at a high price, namely the presence of unitarity-violating ghost states. But if $a_2^\dagger(\bar{k})$ is the Hermitian conjugate of $a_2(\bar{k})$ then $\langle\Omega|a_2(\bar{k})a_2^\dagger(\bar{k})|\Omega\rangle$ would have to be positive. Thus despite the dagger notation $a_2^\dagger(\bar{k})$ could not be the Hermitian conjugate of $a_2(\bar{k})$.

Hence our starting assumption that $\phi(x)$ is Hermitian could not be valid. Consequently, the Hamiltonian that is built out of the $\phi(x)$ field could not be Hermitian either. And in fact we have actually established that it is not, since the diverging of $\psi_0(z(\bar{k}), x(\bar{k}))$ at large $z(\bar{k})$ means that in an integration by parts we could not drop surface terms, with the presence of such surface terms preventing Hermiticity or self-adjointness. With the eigenstates of the Hamiltonian not being normalizable, there not only are negative norm states present, they are infinitely negative.

Surprisingly, it is this very inability to drop surface terms in an integration by parts that actually saves the theory (Bender and Mannheim 2008). Specifically, we have seen that we are working with a Hamiltonian H_S (and likewise $H_{\rm PU}$) that is not Hermitian. However, all the energy eigenvalues associated with H_S and $H_{\rm PU}$ are real. Now Hermiticity is only **sufficient** for real eigenvalues, with the **necessary** condition (Bender and Mannheim 2012, Mannheim 2018) being that the Hamiltonian have an antilinear symmetry. The theory thus falls into the class of PT theories (P is the linear parity operator and T is the antilinear time reversal operator) developed by Bender and collaborators (Bender and Boettcher 1998, Bender 2007, Bender 2019).

Critical to the PT program is that the wave functions be normalizable in some domain in the complex plane, a domain known technically as a Stokes wedge. Since the wave functions are not normalizable with real z or real $z(\bar{k})$, we have to continue z and $z(\bar{k})$ into the complex plane in order to make them normalizable. Then the theory is well-defined, with, as we discuss below, the domain of the measure needed for the path integral accordingly also having to be continued into the complex plane in order to make it be well-defined too (Bender and Mannheim 2008, Mannheim 2018). For the particular case of $\psi_0(z,x)$ and $\psi_0(z(\bar{k}),x(\bar{k}))$, replacing z by $-iz$ and $z(\bar{k})$ by $-iz(\bar{k})$ would then make both $\psi_0(z,x)$ and $\psi_0(z(\bar{k}),x(\bar{k}))$ normalizable. (We have no need to modify x or $x(\bar{k})$ since the wave functions already are well behaved when these quantities become large.)

To achieve the continuation of z or $z(\bar{k})$ at the level of operators we effect similarity transformations as they preserve both energy eigenvalues and canonical commutators. We introduce

$$S({\rm PU})=e^{\pi p_z z/2},\qquad S(S)=e^{\pi\int d^3x\pi_0(\bar{x},t)\phi(\bar{x},t)/2},\tag{7.1}$$

and obtain

$$\begin{aligned}&S({\rm PU})zS({\rm PU})^{-1}=-iz\equiv y,\qquad S({\rm PU})p_zS({\rm PU})^{-1}=ip_z\equiv q,\\ &S(S)z(\bar{k})S(S)^{-1}=-iz(\bar{k})\equiv y(\bar{k}),\qquad S(S)p_z(\bar{k})S(S)^{-1}=ip_z(\bar{k})\equiv q(\bar{k}).\end{aligned}\tag{7.2}$$

7.1 The PU case

For the PU oscillator this leads to

$$
\begin{aligned}
S(\mathrm{PU})H_{\mathrm{PU}}S(\mathrm{PU})^{-1} = \bar{H}_{\mathrm{PU}} = \tfrac{1}{2}p_x^2(t) - iq(t)x(t) + \tfrac{1}{2}\left(\omega_1^2+\omega_2^2\right)x^2(t) + \tfrac{1}{2}\omega_1^2\omega_2^2 y^2(t), \\
[y(t),q(t)] = i, \qquad [x(t),p_x(t)] = i.
\end{aligned}
\tag{7.3}
$$

With p and q being taken to be PT even and y and x being taken to be PT odd (Bender and Mannheim 2008), the PT invariance of $\bar{H}_{\mathrm{PU}}$ and of the $[y,q]=i$ and $[x,p_x]=i$ commutators follows. Now when a Hamiltonian is not Hermitian the action of it to the right and the action of it to the left are not related by Hermitian conjugation. Thus in general one must distinguish between right and left eigenstates, both for the vacuum and the states that can be excited out of it. Thus we represent the $[y,q]=i$ and $[x,p_x]=i$ commutators by $q=-i\overrightarrow{\partial_y}$, $p_x=-i\overrightarrow{\partial_x}$ when acting to the right, and by $q=i\overleftarrow{\partial_y}$, $p_x=i\overleftarrow{\partial_x}$ when acting to the left. This then leads to right and left ground state wave functions of the form (Bender and Mannheim 2008)

$$
\begin{aligned}
\psi_0^R(y,x) &= \exp[-\tfrac{1}{2}(\omega_1+\omega_2)\omega_1\omega_2 y^2 - \omega_1\omega_2 yx - \tfrac{1}{2}(\omega_1+\omega_2)x^2], \\
\psi_0^L(y,x) &= \exp[-\tfrac{1}{2}(\omega_1+\omega_2)\omega_1\omega_2 y^2 + \omega_1\omega_2 yx - \tfrac{1}{2}(\omega_1+\omega_2)x^2], .
\end{aligned}
\tag{7.4}
$$

Given these wave functions the vacuum normalization is given by (Bender and Mannheim 2008)

$$
\begin{aligned}
\langle\Omega^L|\Omega^R\rangle &= \int_{-\infty}^{\infty}dy\int_{-\infty}^{\infty}dx\langle\Omega^L|y,x\rangle\langle y,x|\Omega^R\rangle = \int_{-\infty}^{\infty}dy\int_{-\infty}^{\infty}dx\psi_0^L(y,x)\psi_0^R(y,x) \\
&= \int_{-\infty}^{\infty}dy\int_{-\infty}^{\infty}dx\exp[-(\omega_1+\omega_2)\omega_1\omega_2y^2-(\omega_1+\omega_2)x^2] = \frac{\pi}{(\omega_1\omega_2)^{1/2}(\omega_1+\omega_2)},
\end{aligned}
\tag{7.5}
$$

with the vacuum state thus being normalizable. In the following we shall understand the wave functions to have been normalized to one, so that $\int dydx\psi_0^L(y,x)\psi_0^R(y,x)=1$ and $\langle\Omega^L|\Omega^R\rangle=1$.

With the above PT assignments and with $\dot{y} = i[\bar{H}_{\rm PU}, y] = -ix$, $\dot{x} = p_x$, $\dot{p}_x = iq - (\omega_1^2 + \omega_2^2)x$, $\dot{q} = -\omega_1^2\omega_2^2 y$, we set

$$
\begin{aligned}
y(t) &= -ia_1 e^{-i\omega_1 t} + a_2 e^{-i\omega_2 t} - i\hat{a}_1 e^{i\omega_1 t} + \hat{a}_2 e^{i\omega_2 t}, \\
x(t) &= -i\omega_1 a_1 e^{-i\omega_1 t} + \omega_2 a_2 e^{-i\omega_2 t} + i\omega_1 \hat{a}_1 e^{i\omega_1 t} - \omega_2 \hat{a}_2 e^{i\omega_2 t}, \\
p_x(t) &= -\omega_1^2 a_1 e^{-i\omega_1 t} - i\omega_2^2 a_2 e^{-i\omega_2 t} - \omega_1^2 \hat{a}_1 e^{i\omega_1 t} - i\omega_2^2 \hat{a}_2 e^{i\omega_2 t}, \\
q(t) &= \omega_1\omega_2[-\omega_2 a_1 e^{-i\omega_1 t} - i\omega_1 a_2 e^{-i\omega_2 t} + \omega_2 \hat{a}_1 e^{i\omega_1 t} + i\omega_1 \hat{a}_2 e^{i\omega_2 t}], \\
a_1 e^{-i\omega_1 t} &= \frac{1}{2(\omega_1^2 - \omega_2^2)}\left[-i\omega_2^2 y(t) - p_x(t) + i\omega_1 x(t) + \frac{q(t)}{\omega_1}\right], \\
\hat{a}_1 e^{+i\omega_1 t} &= \frac{1}{2(\omega_1^2 - \omega_2^2)}\left[-i\omega_2^2 y(t) - p_x(t) - i\omega_1 x(t) - \frac{q(t)}{\omega_1}\right], \\
ia_2 e^{-i\omega_2 t} &= \frac{1}{2(\omega_1^2 - \omega_2^2)}\left[i\omega_1^2 y(t) + p_x(t) - i\omega_2 x(t) - \frac{q(t)}{\omega_2}\right], \\
i\hat{a}_2 e^{+i\omega_2 t} &= \frac{1}{2(\omega_1^2 - \omega_2^2)}\left[i\omega_1^2 y(t) + p_x(t) + i\omega_2 x(t) + \frac{q(t)}{\omega_2}\right].
\end{aligned}
\tag{7.6}
$$

In (7.6) we have introduced a_1, a_2, $\hat{a}_1$ and $\hat{a}_2$, with the four creation and annihilation operators obeying $PTa_1TP = a_1$, $PTa_2TP = -a_2$, $PT\hat{a}_1TP = \hat{a}_1$, $PT\hat{a}_2TP = -\hat{a}_2$, so as to enforce the PT assignments of y, x, p_x and q. Comparing with (5.7) we have $(a_1, a_2, a_1^\dagger, a_2^\dagger) \to (a_1, ia_2, \hat{a}_1, i\hat{a}_2)$.

With (7.3) and (7.6) the Hamiltonian is given by

$$\bar{H}_{\rm PU} = 2(\omega_1^2 - \omega_2^2)\left(\omega_1^2 \hat{a}_1 a_1 + \omega_2^2 \hat{a}_2 a_2\right) + \tfrac{1}{2}(\omega_1 + \omega_2), \tag{7.7}$$

and the operator commutation algebra is given by

$$\begin{aligned} &[a_1, \hat{a}_1] = \frac{1}{2\omega_1(\omega_1^2 - \omega_2^2)}, \quad [a_2, \hat{a}_2] = \frac{1}{2\omega_2(\omega_1^2 - \omega_2^2)}, \\ &[a_1, a_2] = 0, \quad [a_1, \hat{a}_2] = 0, \quad [\hat{a}_1, a_2] = 0, \quad [\hat{a}_1, \hat{a}_2] = 0. \end{aligned} \tag{7.8}$$

With the PT assignments of a_1, a_2, $\hat{a}_1$ and $\hat{a}_2$, we confirm the PT invariance of (7.7) and (7.8). In (7.7) and (7.8) the relative signs are all positive (we take $\omega_1 > \omega_2 > 0$ for definitiveness), so these equations define a standard positive energy, positive norm, two-dimensional harmonic oscillator system. Given the creation and annihilation operators the left and right vacua are defined by

$$\langle \Omega^L | \hat{a}_1 = 0, \qquad \langle \Omega^L | \hat{a}_2 = 0, \qquad a_1 | \Omega^R \rangle = 0, \qquad a_2 | \Omega^R \rangle = 0. \tag{7.9}$$

By exciting modes out of the left and right vacua we can build excited states that have positive norm (Bender and Mannheim 2008), viz. $\langle n^L | m^R \rangle = \delta_{nm}$, and obey a completeness relation

$$\sum |n_1^R\rangle\langle n_1^L| + \sum |n_2^R\rangle\langle n_2^L| = I. \tag{7.10}$$

Even though these norms are all positive, the insertion of (7.10) into $-i\langle \Omega^L | T[y(t)y(0)] | \Omega^R \rangle$ (corresponding to $+i\langle \Omega^L | T[z(t)z(0)] | \Omega^R \rangle$) generates the relative minus sign in the nonrelativistic limit of the $-[1/(k^2 - M_1^2) - 1/(k^2 - M_2^2)]/(M_1^2 - M_2^2)$ propagator given in (4.2), viz. $-[1/(\omega^2 - \omega_1^2) - 1/(\omega^2 - \omega_2^2)]/(\omega_1^2 - \omega_2^2)$. We thus establish the consistency and physical viability of the similarity transformed PU oscillator theory.

7.2 Transforming to a Hermitian Hamiltonian

While not obeying $H_{ij}^* = H_{ji}$ in the basis of its eigenfunctions, the $\bar{H}_{PU}$ Hamiltonian given in (7.3) is self-adjoint and has all eigenvalues real. In addition its eigenspectrum is complete (a complete set of polynomial functions of x and y times the ground state wave functions given in (7.4)(Bender and Mannheim 2008). Thus by a similarity transformation $\bar{H}_{PU}$ can be brought to a basis in which $H_{ij}^* = H_{ji}$. (In general if $H' = H'^\dagger$ then under a similarity but not unitary transformation of the form $H' = SHS^{-1}$ we have $SHS^{-1} = S^{-1\dagger}H^\dagger S^\dagger$, i.e, $H^\dagger = S^\dagger SH(S^\dagger S)^{-1}$, with the relation $H' = H'^\dagger$ not being invariant under a non-unitary similarity transformation.) As shown in (Bender and Mannheim 2008) for $H_{\rm PU}$ we introduce

$$Q = \alpha p_x q + \alpha\omega_1^2\omega_2^2 xy, \qquad \alpha = \frac{1}{\omega_1\omega_2}\log\left(\frac{\omega_1+\omega_2}{\omega_1-\omega_2}\right), \tag{7.11}$$

with the requisite transformation then being given by

$$\begin{aligned}
e^{-Q/2}ye^{Q/2} &= y\cosh\theta + i(\omega_1\omega_2)^{-1}p_x\sinh\theta,\\
e^{-Q/2}xe^{Q/2} &= x\cosh\theta + i(\omega_1\omega_2)^{-1}q\sinh\theta,\\
e^{-Q/2}pe^{Q/2} &= p_x\cosh\theta - i(\omega_1\omega_2)y\sinh\theta,\\
e^{-Q/2}qe^{Q/2} &= q\cosh\theta - i(\omega_1\omega_2)x\sinh\theta,\\
e^{-Q/2}\bar{H}_{PU}e^{Q/2} &= \bar{H}'_{PU} = \frac{p_x^2}{2} + \frac{q^2}{2\omega_1^2} + \frac{1}{2}\omega_1^2x^2 + \frac{1}{2}\omega_1^2\omega_2^2y^2,
\end{aligned} \tag{7.12}$$

where $\theta = \alpha\omega_1\omega_2/2 = \operatorname{arc\,sinh}[\omega_2/(\omega_1^2-\omega_2^2)^{1/2}]$. We recognize $\bar{H}'_{\rm PU}$ as being a fully acceptable standard, positive norm two-dimensional oscillator system, one for which we can use the Dirac inner product. Moreover, by the analysis given for the harmonic oscillator in Sec. 2, it follows that in this basis the Dirac norm of the vacuum is finite. Similarly, by extension, following an analogous transformation for each $\bar{k}$, so is the vacuum Dirac norm of the second-order-derivative plus fourth-order derivative scalar field theory.

In addition we note that with its phase being $-Q/2$ rather than $-iQ/2$, the $e^{-Q/2}$ operator is not unitary. The transformation from $\bar{H}_{PU}$ to $\bar{H}'_{PU}$ is thus not a unitary transformation, but is a transformation from a skew basis with eigenvectors $|n\rangle$ to an orthogonal basis with eigenvectors

$$|n'\rangle = e^{-Q/2}|n\rangle, \qquad \langle n'| = \langle n|e^{-Q/2}. \tag{7.13}$$

Then since $\langle n'|m'\rangle = \delta_{mn}$, the eigenstates of $\bar{H}$ obey

$$\begin{aligned}
&\langle n|e^{-Q}|m\rangle = \delta_{mn}, \quad \sum_n |n\rangle\langle n|e^{-Q} = I, \\
&\bar{H} = \sum_n |n\rangle E_n \langle n|e^{-Q}, \quad \bar{H}|n\rangle = E_n|n\rangle, \quad \langle n|e^{-Q}\bar{H} = \langle n|e^{-Q}E_n.
\end{aligned} \tag{7.14}$$

We thus recognize the inner product as being not $\langle n|m\rangle$ but $\langle n|e^{-Q}|m\rangle$, with the conjugate of $|n\rangle$ being $\langle n|e^{-Q}$. This state is also the PT conjugate of $|n\rangle$, so that the inner product is the overlap of a state with its PT conjugate rather than that with its Hermitian conjugate, just as we had noted earlier. And as such this inner product is positive definite since $\langle n'|m'\rangle = \delta_{mn}$ is. The PU oscillator theory (and by analog the scalar quantum field theory) is thus a fully viable unitary theory. Thus starting from H_{PU} given in (5.3) we only need make two similarity transformations, viz. (7.2) and (7.12), in order to be able to establish that the theory is free of negative norm states, and has a vacuum with a finite and positive norm. From the form given in (7.12) for $\bar{H}'_{PU}$ it follows that all the operators in it are observable quantum operators, with all experimental measurements then only involving the real quantities that are their eigenvalues.

7.3 The relativistic case

For $S(S)H_SS(S)^{-1} = \bar{H}_S$ we introduce creation and annihilation operators for $\bar{\phi} = S(S)\phi S(S)^{-1} = -i\phi(x)$ of the form

$$\bar{\phi}(x) = \int \frac{d^3k}{(2\pi)^{3/2}} \left[-ia_1(\bar{k})e^{-i\omega_1(\bar{k})t+i\bar{k}\cdot\bar{x}} + a_2(\bar{k})e^{-i\omega_2(\bar{k})t+i\bar{k}\cdot\bar{x}} - i\hat{a}_1(\bar{k})e^{i\omega_1(\bar{k})t-i\bar{k}\cdot\bar{x}} + \hat{a}_2(\bar{k})e^{i\omega_2(\bar{k})t-i\bar{k}\cdot\bar{x}}\right]. \tag{7.15}$$

Comparing with (4.5) we have $(a_1(\bar{k}), a_2(\bar{k}), a_1^\dagger(\bar{k}), a_2^\dagger(\bar{k})) \to (a_1(\bar{k}), ia_2(\bar{k}), \hat{a}_1(\bar{k}), i\hat{a}_2(\bar{k}))$. Unlike $y(t)$, $\bar{\phi}(x)$ is PT even. The PT even Hamiltonian and PT-preserving commutation relations are given by (Bender and Mannheim 2008)

$$\begin{aligned} S(S)H_SS(S)^{-1} = \bar{H}_S \;&=\; \frac{1}{2}\int d^3k\Big[2(M_1^2 - M_2^2)(\bar{k}^2 + M_1^2)\left[\hat{a}_1(\bar{k})a_1(\bar{k}) + a_1(\bar{k})\hat{a}_1(\bar{k})\right] \\ &+\; 2(M_1^2 - M_2^2)(\bar{k}^2 + M_2^2)\left[\hat{a}_2(\bar{k})a_2(\bar{k}) + a_2(\bar{k})\hat{a}_2(\bar{k})\right]\Big], \end{aligned} \tag{7.16}$$

and

$$\begin{aligned} &[\dot{\bar{\phi}}(\bar{x},t), \bar{\phi}(0)] = 0, \qquad [\ddot{\bar{\phi}}(\bar{x},t), \bar{\phi}(0)] = 0, \qquad [\dddot{\bar{\phi}}(\bar{x},t), \bar{\phi}(0)] = i\delta^3(x), \\ &[a_1(\bar{k}), \hat{a}_1(\bar{k}')] = [2(M_1^2 - M_2^2)(\bar{k}^2 + M_1^2)^{1/2}]^{-1}\delta^3(\bar{k} - \bar{k}'), \\ &[a_2(\bar{k}), \hat{a}_2(\bar{k}')] = [2(M_1^2 - M_2^2)(\bar{k}^2 + M_2^2)^{1/2}]^{-1}\delta^3(\bar{k} - \bar{k}'), \\ &[a_1(\bar{k}), a_2(\bar{k}')] = 0, \quad [a_1(\bar{k}), \hat{a}_2(\bar{k}')] = 0, \quad [\hat{a}_1(\bar{k}), a_2(\bar{k}')] = 0, \quad [\hat{a}_1(\bar{k}), \hat{a}_2(\bar{k}')] = 0. \end{aligned} \tag{7.17}$$

With all relative signs being positive (we take $M_1^2 > M_2^2$ for definitiveness), there are no states of negative norm or of negative energy. The discussion completely parallels that of the PU oscillator model given above.

We introduce

$$
\begin{aligned}
y(\bar{k},t) &= -ia_1(\bar{k})e^{-i\omega_1(\bar{k})t} + a_2(\bar{k})e^{-i\omega_2(\bar{k})t} - i\hat{a}_1(\bar{k})e^{i\omega_1(\bar{k})t} + \hat{a}_2(\bar{k})e^{i\omega_2(\bar{k})t},\\
x(\bar{k},t) &= -i\omega_1(\bar{k})a_1(\bar{k})e^{-i\omega_1(\bar{k})t} + \omega_2(\bar{k})a_2(\bar{k})e^{-i\omega_2(\bar{k})t} + i\omega_1(\bar{k})\hat{a}_1(\bar{k})e^{i\omega_1(\bar{k})t} - \omega_2(\bar{k})\hat{a}_2(\bar{k})e^{i\omega_2(\bar{k})t},\\
p_x(\bar{k},t) &= -\omega_1^2(\bar{k})a_1(\bar{k})e^{-i\omega_1(\bar{k})t} - i\omega_2^2(\bar{k})a_2(\bar{k})e^{-i\omega_2(\bar{k})t} - \omega_1^2(\bar{k})\hat{a}_1(\bar{k})e^{i\omega_1(\bar{k})t} - i\omega_2^2(\bar{k})\hat{a}_2(\bar{k})e^{i\omega_2(\bar{k})t},\\
q(\bar{k},t) &= \omega_1(\bar{k})\omega_2(\bar{k})[-\omega_2(\bar{k})a_1(\bar{k})e^{-i\omega_1(\bar{k})t} - i\omega_1(\bar{k})a_2(\bar{k})e^{-i\omega_2(\bar{k})t} + \omega_2(\bar{k})\hat{a}_1(\bar{k})e^{i\omega_1(\bar{k})t} + i\omega_1(\bar{k})\hat{a}_2(\bar{k})e^{i\omega_2(\bar{k})t}],
\end{aligned}
\tag{7.18}
$$

with $\bar{H}_S$ then taking the form

$$
\bar{H}_S = \int d^3k \left[\frac{p_x^2(\bar{k},t)}{2} - iq(\bar{k},t)x(\bar{k},t) + \frac{1}{2}\left[\omega_1^2(\bar{k}) + \omega_2^2(\bar{k})\right] x^2(\bar{k},t) + \frac{1}{2}\omega_1^2(\bar{k})\omega_2^2(\bar{k})y^2(\bar{k},t)\right]. \tag{7.19}
$$

Introducing left and right vacua that obey

$$
\langle\Omega^L|\hat{a}_1(\bar{k}) = 0, \qquad \langle\Omega^L|\hat{a}_2(\bar{k}) = 0, \qquad a_1(\bar{k})|\Omega^R\rangle = 0, \qquad a_2(\bar{k})|\Omega^R\rangle = 0 \tag{7.20}
$$

for all $\bar{k}$, we find that

$$
\begin{aligned}
\langle\Omega^L|\bar{H}_S|\Omega^R\rangle &= \int d^3k\left[\frac{1}{2}(\bar{k}^2+M_1^2)^{1/2} + \frac{1}{2}(\bar{k}^2+M_2^2)^{1/2}\right]\delta^3(0),\\
\langle\Omega^L|\Omega^R\rangle &= \Pi_{\bar{k}}\int_{-\infty}^{\infty} dy(\bar{k})\int_{-\infty}^{\infty} dx(\bar{k})\langle\Omega^L|y(\bar{k}),x(\bar{k})\rangle\langle y(\bar{k}),x(\bar{k})|\Omega^R\rangle\\
&= \Pi_{\bar{k}}\int_{-\infty}^{\infty} dy(\bar{k})\int_{-\infty}^{\infty} dx(\bar{k})\psi_0^L(y(\bar{k}),x(\bar{k}))\psi_0^R(y(\bar{k}),x(\bar{k})) = \Pi_{\bar{k}}1 = 1.
\end{aligned}
\tag{7.21}
$$

We thus confirm that the vacuum normalization is both finite and positive, while the vacuum energy has the conventional zero-point infinity associated with an infinite number of modes. (This infinity occurs because $\bar{H}_S$ contains an infinite number of modes and not because $\langle\Omega^L|\Omega^R\rangle$ itself is infinite.) We thus establish the consistency and physical viability of the similarity transformed higher-derivative scalar field theory. And we note that even though all the norms are positive, the insertion of (7.10) into $-i\langle\Omega^L|T[\bar{\phi}(x)\bar{\phi}(0)]|\Omega^R\rangle$ (corresponding to $+i\langle\Omega^L|T[\phi(x)\phi(0)]|\Omega^R\rangle$) generates the relative minus sign in $-[1/(k^2-M_1^2)-1/(k^2-M_2^2)]/(M_1^2-M_2^2)$ (Bender and Mannheim 2008). Thus with one similarity transform into an appropriate Stokes wedge we solve both the vacuum normalization problem and the negative norm problem.

At this point we can see the key aspect of our study. Ordinarily in quantum field theory it is taken as a given that one should use the Dirac inner product $\langle\Omega|\Omega\rangle$, viz. $\langle\Omega^R|\Omega^R\rangle$, for the vacuum. And also it is taken as a given that this inner product is finite. In this paper we have provided a procedure for checking whether this is in fact the case, and presented a second-order plus fourth-order derivative model in which it explicitly is not finite. For this particular model we have found a different inner product, viz. $\langle\Omega^L|\Omega^R\rangle$, that is finite. (For a Hamiltonian that is Hermitian $|\Omega^R\rangle=|\Omega\rangle$, $\langle\Omega^L|=\langle\Omega|$.) And thus in general one has to determine whether or not $\langle\Omega^R|\Omega^R\rangle$ is finite on case by case basis. We now discuss our findings from the perspective of path integrals.

8 Path integrals and the normalization of the vacuum

The Minkowski path integral associated with the field theory action given in (4.1) is of the form

$$PI(MINK) = \int D[\phi]D[\sigma_\mu]\exp\left[\frac{i}{2}\int_{-\infty}^{\infty} d^4x\left[\partial_\nu\sigma_\mu\partial^\nu\sigma^\mu - \left(M_1^2 + M_2^2\right)\sigma_\mu\sigma^\mu + M_1^2M_2^2\phi^2\right]\right], \tag{8.1}$$

where $\sigma_\mu = \partial_\mu\phi$. Since the theory is fourth order we need four pieces of information to solve the equations of motion. The pieces that are the most convenient for path integral purposes are two initial and two final conditions, hence the path integral measure is over both ϕ and σ_μ.

In order to damp out oscillations we choose the Feynman $i\epsilon$ prescription in which we replace M_1^2 and M_2^2 by $M_1^2 - i\epsilon$, $M_2^2 - i\epsilon$. For the path integral this yields

$$PI(MINK) = \int D[\phi]D[\sigma_\mu]\exp\left[\frac{1}{2}\int_{-\infty}^{\infty} d^4x\left[i\partial_\nu\sigma_\mu\partial^\nu\sigma^\mu - i\left(M_1^2 + M_2^2\right)\sigma_\mu\sigma^\mu + iM_1^2M_2^2\phi^2 - 2\epsilon\sigma_\mu\sigma^\mu + (M_1^2 + M_2^2)\epsilon\phi^2\right]\right]. \tag{8.2}$$

With ϕ and σ_μ being taken to be real and with $\sigma_\mu\sigma^\mu$ being taken to be timelike on every path, the σ_μ path integration is damped but the ϕ path integration is not. Consequently, as integrated with a real measure the path integral does not exist. Now the path integral is used to generate time-ordered Green's functions such as $D(x) = i\langle\Omega|T[\phi(x)\phi(0)]|\Omega\rangle$ (hence the $i\epsilon$ prescription). And thus these Green's functions will not be finite, with the vacuum in which the Green's function matrix elements are evaluated thus not being normalizable.

Study of the Minkowski path integral thus gives us an alternate way to determine whether or not $\langle\Omega|\Omega\rangle$ is finite: the path integral with a real measure either exists or does not exist.

For the unconventional $i\epsilon$ prescription in which we replace M_1^2 and M_2^2 by $M_1^2 - i\epsilon$, $M_2^2 + i\epsilon$ the path integral takes the form

$$PI(MINK) = \int D[\phi]D[\sigma_\mu]\exp\left[\frac{1}{2}\int_{-\infty}^{\infty} d^4x\left[i\partial_\nu\sigma_\mu\partial^\nu\sigma^\mu - i\left(M_1^2 + M_2^2\right)\sigma_\mu\sigma^\mu + iM_1^2M_2^2\phi^2 - (M_1^2 - M_2^2)\epsilon\phi^2\right]\right], \tag{8.3}$$

and has no damping on the σ_μ path integration at all. The unconventional $i\epsilon$ prescription for the Feynman contour that leads to an unbounded from below energy spectrum thus cannot be associated with a well-defined path integral, and we cannot consider it further.

Thus the only Feynman $i\epsilon$ prescription that can be relevant is the standard one with $M_1^2 - i\epsilon$, $M_2^2 - i\epsilon$. However even with this choice the ϕ path integration is not damped if ϕ is real. It becomes damped if we do not require ϕ to be real, but instead take it to be pure imaginary (though $(\mathrm{Im}[\phi])^2 > (\mathrm{Re}[\phi])^2$ would suffice). With $\bar{\phi} = -i\phi$ we replace (8.2) by

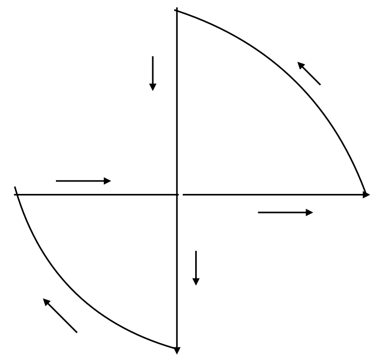
$$PI(MINK) = \int D[\bar{\phi}]D[\sigma_\mu] \exp\left[\frac{1}{2}\int_{-\infty}^{\infty} d^4x \left[i\partial_\nu\sigma_\mu\partial^\nu\sigma^\mu - i\left(M_1^2 + M_2^2\right)\sigma_\mu\sigma^\mu - iM_1^2M_2^2\bar{\phi}^2 \right.\right.$$
$$\left.\left. - 2\epsilon\sigma_\mu\sigma^\mu - (M_1^2 + M_2^2)\epsilon\bar{\phi}^2\right]\right]. \tag{8.4}$$

With $\bar{\phi}$ and σ_μ being taken to be real and with $\sigma_\mu\sigma^\mu$ being taken to be timelike on every path, the path integral is now well defined and the theory is consistent. (In a study of quantum gravity 't Hooft 2011 has also suggested that the path integral measure be continued into the complex domain.)

This puts us into a domain in the complex plane (known as a Stokes wedge) in which the path integral is now fully defined, and now the vacuum state is normalizable. This completely parallels the discussion of $\psi_0(z,x)$ that we gave above.

The higher-derivative quantum scalr field theory has two distinct realizations, the Ostrogradski one and the Feynman $i\epsilon$ prescription. Now it cannot be the case that one and the same path integral has two totally different realizations, and so we need differentiate between them. This is done by having them be defined with differing variables. Thus for Ostrogradski realization ϕ is real, while for the Feynman $i\epsilon$ prescription ϕ is pure imaginary. These two options for ϕ then lead to two completely different theories. However in the end in the Feynman $i\epsilon$ prescription case we still end up with a real $\bar{\phi} = -i\phi$, just as needed to make the path integral exist, and to obtain a real output classical theory. The classical variables are the eigenvalues of the quantum operators once the quantum operators are self-adjoint, and these classical variables are real because of the PT symmetry of the quantum theory.

The Euclidean Path Integral



Our concerns here could be missed in a Euclidean time path integral approach. Specifically, if we disperse in t (assuming of course that we can, i.e., that the Cauchy-Riemann equations for complex t are obeyed), we can write

$$\int_{-\infty}^{\infty} + \int_{\infty}^{i\infty} + \int_{i\infty}^{-i\infty} + \int_{-i\infty}^{-\infty} = \text{pole terms plus cut contributions}, \tag{8.5}$$

i.e., along the real axis, then upper-half-plane quarter circle, then down the imaginary axis, and then lower-half-plane quarter circle. Assuming no pole, cut or circle contributions, and on setting $\tau = it$ and letting I denote the action we obtain

$$I(MINK, z, x) \equiv \int_{-\infty}^{\infty} dt \equiv -\int_{i\infty}^{-i\infty} dt \equiv I(EUCL, z, x),$$
$$PI(EUCL, z, x) = \int D[\phi]D[\sigma_\mu] \exp\left[-\frac{1}{2}\int_{-\infty}^{\infty} d\tau d^3x \left[\partial_\nu\sigma_\mu\partial^\nu\sigma^\mu - \left(M_1^2 + M_2^2\right)\sigma_\mu\sigma^\mu + M_1^2M_2^2\phi^2\right]\right]. \tag{8.6}$$

where now $\sigma_\mu\sigma^\mu$ is spacelike.

Given the overall minus sign that multiplies the Euclidean action on every path, we see that the Euclidean path integral is well behaved (Hawking and Hertog 2002). However, the Minkowski time path integral with a real measure is not.

Thus we conclude that the pole and/or cut and/or circle contributions are not only not ignorable, they generate an infinite contribution. Hence their contribution in a Wick rotation cannot be ignored and the Euclidean time path integral does not correctly describe the situation, and thus we see that even if finite, a Euclidean time path integral approach is only valid if the vacuum state of the theory (as determined in a Minkowski time analysis) is normalizable. Otherwise the Wick contour rotation fails.

9 Implications for radiative corrections in quantum Einstein gravity

As a quantum theory the standard second-order-derivative Einstein gravitational theory with its $1/k^2$ propagator is not renormalizable. Since graviton loops generate higher-derivative gravity terms, one can construct a candidate theory of quantum gravity by augmenting the Einstein Ricci scalar action with a term that is quadratic in the Ricci scalar. This gives a much studied quantum gravity action of the generic form

$$I_{\rm GRAV}=\int d^4x(-g)^{1/2}\left[6M^2R^{\alpha}{}_{\alpha}+(R^{\alpha}{}_{\alpha})^2\right], \tag{9.1}$$

and it can be considered to be an ultraviolet completion of Einstein gravity. This same action also appears in Starobinsky's inflationary universe model.

On adding on a matter source with energy-momentum tensor $T_{\mu\nu}$, variation of this action with respect to the metric generates a gravitational equation of motion of the form

$$-6M^2G^{\mu\nu}+V^{\mu\nu}=-\frac{1}{2}T^{\mu\nu}. \tag{9.2}$$

Here $G_{\mu\nu}$ and $V_{\mu\nu}$ are of the form

$$\begin{aligned} G^{\mu\nu} &= R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}g^{\alpha\beta}R_{\alpha\beta}, \\ V^{\mu\nu} &= 2g^{\mu\nu}\nabla_{\beta}\nabla^{\beta}R^{\alpha}{}_{\alpha}-2\nabla^{\nu}\nabla^{\mu}R^{\alpha}{}_{\alpha}-2R^{\alpha}{}_{\alpha}R^{\mu\nu}+\frac{1}{2}g^{\mu\nu}(R^{\alpha}{}_{\alpha})^2. \end{aligned} \tag{9.3}$$

If we now linearize about flat spacetime with background metric $\eta_{\mu\nu}$ and fluctuation metric $g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$, to first perturbative order we obtain

$$\begin{aligned} \delta G_{\mu\nu} &= \frac{1}{2}\left(\partial_{\alpha}\partial^{\alpha}h_{\mu\nu}-\partial_{\mu}\partial^{\alpha}h_{\alpha\nu}-\partial_{\nu}\partial^{\alpha}h_{\alpha\mu}+\partial_{\mu}\partial_{\nu}h\right)-\frac{1}{2}\eta_{\mu\nu}\left(\partial_{\alpha}\partial^{\alpha}h-\partial^{\alpha}\partial^{\beta}h_{\alpha\beta}\right), \\ \delta V_{\mu\nu} &= [2\eta_{\mu\nu}\partial_{\alpha}\partial^{\alpha}-2\partial_{\mu}\partial_{\nu}][\partial_{\beta}\partial^{\beta}h-\partial_{\lambda}\partial_{\kappa}h^{\lambda\kappa}], \end{aligned} \tag{9.4}$$

where $h=\eta^{\mu\nu}h_{\mu\nu}$. On taking the trace of the fluctuation around a background (9.2) we obtain

$$[M^2+\partial_{\beta}\partial^{\beta}]\left(\partial_{\lambda}\partial^{\lambda}h-\partial_{\kappa}\partial_{\lambda}h^{\kappa\lambda}\right)=-\frac{1}{12}\eta^{\mu\nu}\delta T_{\mu\nu}. \tag{9.5}$$

In the convenient transverse gauge where $\partial_\mu h^{\mu\nu} = 0$, the propagator for h is given by

$$D(h, k^2) = -\frac{1}{k^2(k^2 - M^2)} = \frac{1}{M^2}\left(\frac{1}{k^2} - \frac{1}{k^2 - M^2}\right). \tag{9.6}$$

As we see, in this case the $1/k^2$ graviton propagator for h that would be associated with the Einstein tensor $\delta G_{\mu\nu}$ alone is replaced by a $D(h, k^2) = [1/k^2 - 1/(k^2 - M^2)]/M^2$ propagator. And now the leading behavior at large momenta is $-1/k^4$. In consequence, the theory is thought to be renormalizable (Stelle 1977). But since $\langle\Omega|\Omega\rangle$ is not finite the proof of renormalizability has a flaw in it. Fortunately, the flaw is not fatal, and we rectify it below.

We recognize $D(h, k^2)$ as being of the same form as the second-order plus fourth-order scalar field theory propagator that was given in (4.2), with ϕ being replaced by h and with $M_1^2 = M^2$, $M_2^2 = 0$. We can thus give h an equivalent effective action of the form

$$I_h = \frac{1}{2}\int d^4x \left[\partial_\mu\partial_\nu h \partial^\mu\partial^\nu h - M^2 \partial_\mu h \partial^\mu h\right]. \tag{9.7}$$

The action given in (9.7) thus shares the same vacuum state normalization and negative norm challenges as the scalar field action given in (4.1).

Thus if, as is conventional, we take h to be Hermitian we would immediately encounter the negative norm problem associated with the relative minus sign in (9.6). However, since M^2 is Planck scale in magnitude, this difficulty can be postponed until observations can reach that energy scale. **However, the lack of normalizabilty of the vacuum state has consequences at all energies and cannot be postponed at all**. Specifically, with $\langle\Omega|\Omega\rangle$ being infinite we cannot even identify the propagator as $i\langle\Omega|T[h(x)h(0)]|\Omega\rangle$ since in analog to (1.8) it will obey

$$(\partial_t^2 - \bar{\nabla}^2)(\partial_t^2 - \bar{\nabla}^2 + M^2)D(h, x) = -\langle\Omega|\Omega\rangle\delta^4(x). \tag{9.8}$$

Consequently, we cannot make the standard Wick contraction expansion. And the Feynman rules that are used presupposing that $\langle\Omega|\Omega\rangle$ is finite are therefore not valid. The effective field theory approach to gravity also fails, because even at energies with $k^2 \ll M^2$ the vacuum is still not normalizable.

However, as noted above, we can resolve all of these concerns by dropping the requirement that h be Hermitian, and transform it to $i\bar{h}$, where $\bar{h}$ is self-adjoint in its own eigenstate basis. Then, with the theory being recognized as a PT theory, vacuum state normalization and negative-norm problems are resolved and the theory is consistent. Moreover, the propagator is given by $-i\langle\Omega^L|T[\bar{h}(x)\bar{h}(0)]|\Omega^R\rangle$ (corresponding to $+i\langle\Omega^L|T[h(x)h(0)]|\Omega^R\rangle$). And with the propagator still being given by (9.6) as it satisfies $(\partial_t^2-\bar{\nabla}^2+M^2)(\partial_t^2-\bar{\nabla}^2)[-i\langle\Omega^L|T[\bar{h}(x)\bar{h}(0)]|\Omega^R\rangle]=-\delta^4(x)$, all the steps needed to prove renormalizability are now valid. Consequently, the theory can now be offered as a fully consistent, unitary and renormalizable theory of quantum gravity, and can thus serve as the ultraviolet completion of Einstein gravity

In that case the only concern is that even though the M^2 field now has a finite, positive norm, it still remains in the spectrum and would eventually have to be observed. Also of course the theory shares the dark matter and dark energy problems of its now bona fide low energy effective Einstein theory.

As can be seen from (9.7), the only reason that there is an M^2 term at all is because we are considering an action that has both second-order and fourth-order terms. With a pure fourth-order theory there would be no dimensionful parameter in the action and the theory would be scale invariant. If like the gauge theories of $SU(3)\times SU(2)\times U(1)$ this scale symmetry is also local, we would be led to conformal gravity, a metric theory of gravity in which the action is left invariant under local changes of the metric of the form $g_{\mu\nu}(x)\to e^{2\alpha(x)}g_{\mu\nu}(x)$, where $\alpha(x)$ is a local function of the coordinates. The conformal gravity theory has been advocated and explored in (Mannheim 2006, Mannheim 2017) and references therein. And 't Hooft ('t Hooft 2015) has also argued that there should be an underlying local conformal symmetry in nature.

In the conformal gravity theory an action that is to be a polynomial function of the metric has the unique form

$$I_{\rm W}=-\alpha_g\int d^4x\,(-g)^{1/2}C_{\lambda\mu\nu\kappa}C^{\lambda\mu\nu\kappa}\equiv -2\alpha_g\int d^4x\,(-g)^{1/2}\left[R_{\mu\kappa}R^{\mu\kappa}-\frac{1}{3}(R^{\alpha}{}_{\alpha})^2\right], \tag{9.9}$$

where α_g is a dimensionless gravitational coupling constant, and $C_{\lambda\mu\nu\kappa}$ is the conformal Weyl tensor. The perturbative propagator has a $-1/k^4$ behavior at all k^2, and with its large k^2 behavior the theory is renormalizable (Fradkin and Tseytlin 1985). With a $-1/k^4$ propagator it would initially appear that there would be two massless particles at $k^2=0$. However, we cannot use the partial fraction decomposition given in (9.6) as a guide since its $1/M^2$ prefactor is singular in the $M^2\to 0$ limit. Because of this singular behavior the $M^2=0$ Hamiltonian becomes of nondiagonalizable Jordan-block form and only has one massless eigenstate, with the other would-be massless eigenstate becoming nonstationary (Bender and Mannheim 2008).

Thus the propagator should be constructed not as the $M^2\to 0$ limit of the ghost-like but actually ghost-free (9.6), but instead as the manifestly ghost-free limit

$$-\frac{1}{(k^2+i\epsilon)^2}=-\lim_{M^2\to 0}\frac{d}{dM^2}\left(\frac{1}{k^2-M^2+i\epsilon}\right), \tag{9.10}$$

a limit that shows that there is only one $k^2=0$ pole not two. With the Hamiltonian not being diagonalizable, it could not be Hermitian. It does however have a PT symmetry, with is ground state being normalizable. Conformal gravity is thus a fully consistent theory of quantum gravity, one which despite its fourth-order character possesses no states of negative norm, and only one massless particle, not two.

10 Final Comments

For a quantum field theory to be physically relevant it must be formulatable in a Hilbert space with an inner product that is time independent, finite and nonnegative. However, in and of itself, specifying an action and a set of canonical commutators is not enough to either fix the Hilbert space or specify the appropriate inner product. Ordinarily, one supplements these requirements with the additional (generally regarded as self-evident) requirements that the fields and the Hamiltonian of the theory be Hermitian, and that the inner product be the standard, presumed finite, Dirac $\langle n|n\rangle$ one. However, this is not automatic for any theory, and so one needs to check on a case by case basis. And we have presented a procedure for doing so. The procedure is based on using the occupation number space representation to construct an equivalent wave mechanics representation, from which we can check for the normalizability of the vacuum state, and accordingly of the states that can be excited out of it. An alternative but equivalent approach is to check whether or not the Minkowski time path integral with a real measure exists. If it does not, then the standard Dirac inner product is not finite.

Using the occupation number space representation procedure we have found a case, a second-order plus fourth-order scalar field theory, in which the standard Dirac inner product $\langle n|n\rangle$ actually is not finite. In this example the Minkowski time path integral with a real measure diverges even though the Euclidean time path integral does not. Even though contributions from the Wick rotation contour are ordinarily ignored, in this case they cannot be. Thus the use of a Euclidean time path integral can be misleading. And even if the Euclidean time path integral is well behaved, it only gives a good description of the theory if the Minkowski time path integral is well behaved too. Since $\langle\Omega|\Omega\rangle$ is not finite, use of the standard Feynman rules is not valid, with these rules not only leading to states with negative norm, they lead to states with infinite negative norm. This lack of finiteness means that the Hamiltonian is not self-adjoint when acting on these particular states.

However, the Hamiltonian of the second-order plus fourth-order scalar field theory is PT symmetric, so we can use the techniques of the PT symmetry program and continue the fields and the Hamiltonian in this theory into the complex plane. There is then a domain in the complex plane in which one can define an appropriate time-independent, positive and finite inner product, viz. the $\langle L|R\rangle$ overlap of left-eigenstates and right-eigenstates of the resulting Hamiltonian, with the resulting vacuum state then being normalizable, and with there being no states with negative or infinite $\langle L|R\rangle$ norm. In this complex domain it is the Euclidean time path integral that diverges while the Minkowski time path integral does not. So again there are contributions from the Wick rotation contour. In this complex domain the second-order plus fourth-order scalar field theory is fully consistent, unitary and renormalizable, with this analysis being relevant for the construction of a consistent quantum theory of gravity.