Some results on (quantum) Metric-Affine Gravity

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Notation

Metric-Affine Gravity=MAG

	coefficients	cov. der.	curvature
Independent conn.	$A_\mu{}^a{}_b$	D_μ	$F_{\mu\nu}{}^a{}_b$
LC conn.	$\Gamma_\mu{}^a{}_b$	∇_μ	$R_{\mu\nu}{}^a{}_b$

Also “Gauge theories of Gravity”

What is the gauge group?

Not a physically meaningful question.
Same theory can be presented with different gauge groups.

Normally the gauge transformations are either the diffeomorphisms. (coordinate frames)
or diffeomorphisms *and* local Lorentz transformations (orthonormal frames)

Can enlarge the gauge group to diffeomorphisms *and* local $GL(4)$ transformations (arbitrary frames)
This makes sense because:

- physics does not depend on the choice of frame in the tangent bundle
- in a general MAG the holonomy of the connection is $GL(4)$,

$GL(4)$ formalism

In gravity we have a frame field (local linear bases) $\theta^a{}_\mu$ and a fiber metric g_{ab}. They are nonlinear objects

- metric $g_{ab} \in GL(4)/O(1,3)$, (signature $-,+,+,+$),
- frame field $\theta^a{}_\mu \in GL(4)$, ($\det\theta \neq 0$),

They carry nonlinear realizations of $GL(4)$.
Think of them as Goldstone bosons.

$$g_{\mu\nu}(x) = \theta^a{}_\mu(x)\,\theta^b{}_\nu(x)\,g_{ab}(x)$$

Transformations under $GL(4)$

Linear changes of basis=$GL(4)$ gauge transformations

$$\begin{aligned} \theta &\mapsto \Lambda^{-1}\theta \\ g &\mapsto \Lambda^T g \Lambda \end{aligned}$$

Two "unitary gauges" for $GL(4)$:

- $\theta^a_\mu = \delta^a_\mu$: coordinate frames - breaks $GL(4)$ to $\{\mathbf{1}\}$ metric formulation - breaks $GL(4)$ to $\{\mathbf{1}\}$
- $g_{ab} = \eta_{ab}$: orthonormal frames - breaks $GL(4)$ to $O(1,3)$ vierbein formulation

not enough freedom to fix both.

Torsion, Nonmetricity, Curvature

Let A_{abc} be an independent connection

$$\begin{aligned} T_\mu{}^a{}_\nu &= \partial_\mu \theta^a{}_\nu - \partial_\nu \theta^a{}_\mu + A_\mu{}^a{}_b\, \theta^b{}_\nu - A_\nu{}^a{}_b\, \theta^b{}_\mu \\ Q_{\lambda ab} &= -\partial_\lambda g_{ab} + A_\lambda{}^c{}_a\, g_{cb} + A_\lambda{}^c{}_b\, g_{ac} \\ F_{\mu\nu}{}^a{}_b &= \partial_\mu A_\nu{}^a{}_b - \partial_\nu A_\mu{}^a{}_b + A_\mu{}^a{}_c A_\nu{}^c{}_b - A_\nu{}^a{}_c A_\mu{}^c{}_b \end{aligned}$$

T, Q are the covariant derivatives of the Goldstone bosons

Cartan view of MAG

MAG as a theory of frame, metric, connection.

$$S(\theta^a{}_\mu, g_{ab}, A_\mu{}^a{}_b)$$

Powers of T, Q, F and their D-covariant derivatives.

The Levi-Civita connection

Unique connection with $T = Q = 0$

$$\Gamma_{abc} = \frac{1}{2}\left(E_{cab} + E_{abc} - E_{bac}\right) + \frac{1}{2}\left(C_{abc} + C_{bac} - C_{cab}\right)$$

where $E_{cab} = {\theta^{-1}}_c{}^{\lambda}\,\partial_\lambda g_{ab}$
$C_{abc} = g_{ad}\,\theta^d{}_\lambda\left({\theta^{-1}}_b{}^{\mu}\,\partial_\mu {\theta^{-1}}_c{}^{\lambda} - {\theta^{-1}}_c{}^{\mu}\,\partial_\mu {\theta^{-1}}_b{}^{\lambda}\right)$

Its curvature is the Riemann tensor

$$R_{\mu\nu}{}^a{}_b = \partial_\mu \Gamma_\nu{}^a{}_b - \partial_\nu \Gamma_\mu{}^a{}_b + \Gamma_\mu{}^a{}_c \Gamma_\nu{}^c{}_b - \Gamma_\nu{}^a{}_c \Gamma_\mu{}^c{}_b$$

Distortion, disformation, contortion

Any connection A can be split uniquely in

$$A = \Gamma + \Phi$$

("post–Riemannian decomposition") where

$$\phi_{\alpha\beta\gamma} = L_{\alpha\beta\gamma} + K_{\alpha\beta\gamma} \ ,$$

$$\begin{aligned} L_{\alpha\beta\gamma} &= \frac{1}{2}\left(Q_{\alpha\beta\gamma} + Q_{\gamma\beta\alpha} - Q_{\beta\alpha\gamma}\right) \ , \\ K_{\alpha\beta\gamma} &= \frac{1}{2}\left(T_{\alpha\beta\gamma} + T_{\beta\alpha\gamma} - T_{\alpha\gamma\beta}\right) \ . \end{aligned}$$

conversely

$$T_{\alpha\beta\gamma} = \phi_{\alpha\beta\gamma} - \phi_{\gamma\beta\alpha} \ , \qquad Q_{\alpha\beta\gamma} = \phi_{\alpha\beta\gamma} + \phi_{\alpha\gamma\beta}$$

Einstein view of MAG

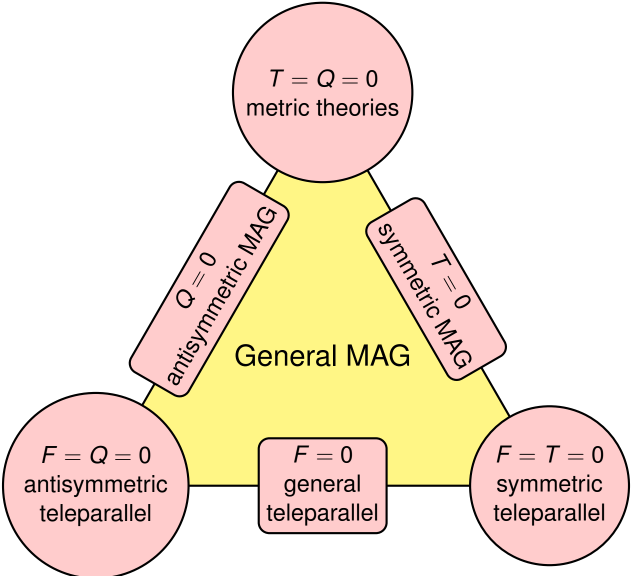
Consequently

$$S(\theta^a{}_\mu, g_{ab}, A_\mu{}^a{}_b) = S(\theta^a{}_\mu, g_{ab}, \Gamma_\mu{}^a{}_b + \phi_\mu{}^a{}_b) = S'(\theta^a{}_\mu, g_{ab}, \Phi_\mu{}^a{}_b)$$

Powers of T, Q, R and their ∇-covariant derivatives.
Now T, Q are independent fields.

Any MAG is equivalent to a *metric* theory of gravity coupled to some "matter" fields.

Main kinematical sublasses of MAGs



Dynamical equivalences: GR

Up to total derivatives, R is equivalent to

$$\mathbb{T} = \frac{1}{4} T_{abc} T^{abc} + \frac{1}{2} T_{abc} T^{acb} - T^{b}{}_{ba} T_{c}{}^{ca}$$

for the antisymmetric MAG,

$$\mathbb{Q} = \frac{1}{4} Q_{abc} Q^{abc} - \frac{1}{2} Q_{abc} Q^{bac} - \frac{1}{4} Q^{b}_{ab} Q^{ac}{}_{c} + \frac{1}{2} Q^{b}_{ab} Q_{c}{}^{ca}$$

for the symmetric MAG and

$$\mathbb{G} = \mathbb{T} + \mathbb{Q} - Q_{abc} T^{abc} - Q_{ab}{}^{b} T_{c}{}^{ca} + Q^{b}{}_{ba} T_{c}{}^{ca} \ .$$

for the general MAG.

Every metric theory has teleparallel equivalents.

Some actions for MAG

Poincaré Gauge Theory (PGT) view $A_{\mu[ab]}$ and $\theta^{a}{}_{\mu}$ as gauge fields
and $F_{\mu\nu[ab]}$ and $T_{\mu}{}^{a}{}_{\nu}$ as field strength.
this motivates

$$\mathcal{L} = a^{TT}TT + c^{FF}FF$$

In teleparallel case $\mathcal{L} = f(\mathbb{T})$ or $f(\mathbb{Q})$.

Let us be systematic and order Lagrangians according to their dimension. (T and Q have dimension one, F and R have dimension 2).

General action of MAG

In Cartan-variables and including terms of dimension 2 and 4:

$$
\begin{aligned}
\mathcal{L} = \; & a^{F} F + a^{TT} TT + a^{TQ} TQ + a^{QQ} QQ \\
& + c^{FF} FF \\
& \quad + c^{FT} FDT + c^{FQ} FDQ + c^{TT} (DT)^2 + c^{TQ} DTDQ + c^{QQ} (DQ)^2 \\
& + c^{FTT} FTT + c^{FTQ} FTQ + c^{FQQ} FQQ \\
& + c^{TTT} TTDT + \ldots + c^{QQQ} QQDQ \\
& + c^{TTTT} TTTT + \ldots + c^{QQQQ} QQQQ \, ,
\end{aligned}
$$

Similar classification in Einstein variables. Large number of invariants.

Counting independent invariants of dimension 4

Antisymmetric (Christensen 1979)

R^2	$(\nabla T)^2$	$R\nabla T$	$R\,T^2$	$T^2\nabla T$	T^4	Total
3	9	2	14	31	33	92

Symmetric

R^2	$(\nabla Q)^2$	$R\nabla Q$	$R\,Q^2$	$Q^2\nabla Q$	Q^4	Total
3	16	4	22	59	69	173

General

R^2	$(\nabla\phi)^2$	$R\nabla\phi$	$R\,\phi^2$	$\phi^2\nabla\phi$	ϕ^4	Total
3	38	6	56	315	504	922

Further kinematical restrictions: Weyl theory as MAG

In a symmetric MAG with

$$Q_{\mu\alpha\beta} = b_\mu g_{\alpha\beta}$$

the quadratic MAG action collapses to

$$S(g) = \int d^4x\sqrt{g}\left[\frac{g_1 + 12g_4}{32\pi G g_4} b_\mu b^\mu + \frac{g_3}{4} F_{\mu\nu}F^{\mu\nu} + \frac{1}{16\pi G}(2\Lambda - R)\right]$$

which, introducing a Stückelberg field χ can be rewritten

$$S = \int d^4x\sqrt{g}\left[\frac{g_1 + 12g_4}{2} D_\mu\chi D^\mu\chi + g_2\chi^4 + \frac{g_3}{4} F_{\mu\nu}F^{\mu\nu} - g_4\chi^2\mathcal{R}^{(s)}\right] .$$

Weyl gauge field+dilaton

or equivalently

$$S=\int d^4x\sqrt{g}\left[\frac{g_1}{2}D_\mu\chi D^\mu\chi+g_2\chi^4+\frac{g_3}{4}F_{\mu\nu}F^{\mu\nu}-g_4\chi^2\mathcal{R}^{(b)}\right] .$$

$$D_\mu\chi=(\partial_\mu+b_\mu)\chi.$$

D.M. Ghilencea, JHEP 03 (2019) 049, arXiv: 1812.08613 [hep-th]
D. Sauro, O. Zanusso, Class.Quant.Grav. 39 (2022) 18, 185001, arXiv: 2203.08692 [hep-th]

Gravitational Higgs mechanism in Cartan view

In Cartan-variables and in arbitrary frames (generic $GL(4)$ gauge)

$$S_G(\theta, g, A) \;=\; m^2 \int d^4x \sqrt{|g|} \left[T_{\dots} T^{\cdots} + Q_{\dots} Q^{\cdots} + T_{\dots} Q^{\cdots} \right]$$

expanding around flat background: $A = 0$, $\theta = \mathbf{1}$, $g = \eta$

$$\begin{aligned} T_\mu{}^a{}_\nu &= A_\mu{}^a{}_\nu - A_\nu{}^a{}_\mu \\ Q_{\mu ab} &= A_{\mu ab} + A_{\mu ba} \end{aligned}$$

kinetic term of Goldstone bosons becomes

$$S_G = m^2 \int d^4x \, \sqrt{|g|} \, A_{\dots} A^{\cdots}$$

In general a non-degenerate quadratic form

The Palatini term

$$S_P(A, g, \theta) = \frac{m_P^2}{2} \int d^4x \, \sqrt{|g|} \, F(A)_{ab}{}^{ab}$$

also gives

$$\frac{m_P^2}{2} \int d^4x \, \sqrt{|g|} \, \left(A_a{}^{ac} A_{bc}{}^{b} - A_{bac} A^{acb} \right)$$

(a degenerate quadratic form)

Gravitational Higgs mechanism in Einstein view

Using the post-Riemannian expansion
$A = \Gamma + \Phi$

$$
\begin{aligned}
T_\mu{}^a{}_\nu &= \Phi_\mu{}^a{}_\nu - \Phi_\nu{}^a{}_\mu \\
Q_{\mu ab} &= \Phi_{\mu ab} + \Phi_{\mu ba}
\end{aligned}
$$

therefore (in any gauge and without assumptions on the background)

$$
S_G = \int d^4x \, \sqrt{|g|} \, \Phi_{\dots} \Phi^{\dots}
$$

Macroscopic gravity

If mass matrix is nondegenerate, assuming all masses are $O(m)$, at $p \ll m$

$$\Phi = 0 \iff T = Q = 0 \iff A = \Gamma(\theta, g)$$

independent of the detailed form of the action.

To summarize

MAG below the Planck scale looks like a low energy EFT where Higgs phenomenon occurs at Planck scale.
Metric is the order parameter.

The conditions $T = 0$ and $Q = 0$ have the same status as $D\varphi = 0$ in superconductivity (Meissner effect) or similar conditions in the EEχPT.

Core questions for quantum theory of spacetime:

- what is the dynamical origin of the Planck scale?
- why is there a nondegenerate metric?
- is the metric fundamental or “emergent”?

Scenarios

Natural to assume that all masses are $m \sim m_P$. No new d.o.f.s in the domain of validity of the EFT. Then MAG as an EFT is essentially indistinguishable from GR as an EFT.

Still has somewhat increased explanatory power: explains dynamically the conditions $T = Q = 0$.

If there are eigenvalues $m \ll m_P$ there may be new propagating d.o.f.s in the domain of validity of the EFT. What kind of states?

Spin$^{\text{parity}}$ states

$\delta g_{\alpha\beta}$ or $\delta\theta_{\alpha\beta}$

	s	a
TT	$2_4^+, 0_5^+$	1_4^+
TL	1_7^-	1_8^-
LL	0_6^+	-

$\delta A_{\alpha\beta\gamma}$ or $\Phi_{\alpha\beta\gamma}$

	ts	hs	ha	ta
TTT	$3^-, 1_1^-$	$2_1^-, 1_2^-$	$2_2^-, 1_3^-$	0^-
$TTL + TLT + LTT$	$2_1^+, 0_1^+$	-	-	1_3^+
$\frac{3}{2}LTT$	-	$2_2^+, 0_2^+$	$1_2^+,$	-
$TTL + TLT - \frac{1}{2}LTT$	-	1_1^+	$2_3^+, 0_3^+$	-
$TLL + LTL + LLT$	1_4^-	1_5^-	1_6^-	-
LLL	0_4^+	-	-	-

Spin projector formalism

The quadratic action for the variables A, h

$$S^{(2)} = \frac{1}{2}\int \frac{d^4q}{(2\pi)^4}\Big(A^{\lambda\mu\nu}\, \mathcal{O}^{(AA)}_{(C)}{}_{\lambda\mu\nu}{}^{\tau\rho\sigma}\, A_{\tau\rho\sigma} + 2A^{\lambda\mu\nu}\, \mathcal{O}^{(Ah)}_{(C)}{}_{\lambda\mu\nu}{}^{\rho\sigma}\, h_{\rho\sigma} + h^{\mu\nu}\, \mathcal{O}^{(hh)}_{(C)}{}_{\mu\nu}{}^{\rho\sigma}\, h_{\rho\sigma} \Big)$$

can be rewritten

$$\frac{1}{2}\int \frac{d^4q}{(2\pi)^4} \sum_{JPij} a_{ij}(J^{\mathcal{P}})\, \Psi P_{ij}(J^{\mathcal{P}})\Psi \ ,$$

where $\Psi = (A, h)$, e.g in the A-A sector

$$\frac{1}{2}\int \frac{d^4q}{(2\pi)^4} \sum_{JPij} a_{ij}(J^{\mathcal{P}})\, A^{\lambda\mu\nu} P_{ij}(J^{\mathcal{P}})_{\lambda\mu\nu}{}^{\tau\rho\sigma} A_{\tau\rho\sigma} \ ,$$

Ghost- and tachyon-free MAGs

Antisymmetric MAG:

E. Sezgin and P. van Nieuwenhuizen, Phys. Rev. D21, (1980) 3269

Y.C. Lin, M.P. Hobson and A.N. Lasenby, arXiv:1812.02675

Symmetric MAG:

R.P. and E. Sezgin, Phys. Rev. D101, (2020) 084040

C. Marzo, Phys.Rev.D 105 (2022) 6, 065017

DIY MAG:

A. Baldazzi, O. Melichev and R.P. Ann.Phys. 438 (2022) 168757

- measure zero in theory space
- consistency only checked at level of free propagators

A six-parameter family of symmetric MAGs

Projective invariance ($\delta A_\lambda{}^a{}_b = v_\lambda \delta^a_b$),
no spin 3^-, only one (massless) spin 2^+

$$
\begin{aligned}
S(g,A) = -\frac{1}{2}\int d^n x\,\sqrt{|g|}\Big[& -\frac{1}{4}(10c_7+3c_{11})F^{\mu\nu\rho\sigma}\Big(F_{\mu\nu\rho\sigma}-2F_{\mu\rho\nu\sigma}\Big) \\
& +2F^{(13)}_{[\mu\nu]}\Big(c_7 F^{(13)\mu\nu}+c_{11}F^{(14)\mu\nu}\Big)-\frac{2}{3}(c_7+4c_{11})\,F^{(14)}_{[\mu\nu]}F^{(14)\mu\nu} \\
& -a_0 F-\frac{1}{16}(15a_0+120a_6+28a_7-22a_8)\,Q^{\rho\mu\nu}Q_{\rho\mu\nu} \\
& +\frac{1}{8}(9a_0+40a_6-28a_7-6a_8)\,Q^{\rho\mu\nu}Q_{\nu\mu\rho} \\
& +a_6\,Q_\mu Q^\mu+a_7\,\widetilde{Q}_\mu\widetilde{Q}^\mu+\frac{1}{72}(a_0+8a_6-4a_7+74a_8)\,Q_\mu\widetilde{Q}^\mu\Big]
\end{aligned}
$$

New ghost and tachyon-free theories

No ghosts and tachyons provided

$$a_0 > 0 \,, \quad 4\,a_0 + 20\,a_6 - 7\,a_7 + 2\,a_8 < 0 \,,$$

$$(a_0 + 8a_6 - 4a_7 + 2a_8)(89a_0 + 520a_6 - 212a_7 + 82a_8) > 0 \,,$$

$$c_{11} < -2c_7 \;\; \text{for} \;\; c_7 \leq 0 \,, \quad \text{or} \quad c_{11} < -\frac{10}{3}c_7 \;\; \text{for} \;\; c_7 > 0 \,.$$

One massless spin 2^+ one massive spin 2^-, 1^+ and 1^- each

In Einstein form, contain no R^2 terms. Non-renormalizable.

Probably radiatively unstable.

Spin 2^-

Antisymmetric MAG

$$\mathcal{L} = b_1^{TT}\Big[-\frac{1}{3}\nabla_\mu K_{\nu\rho\sigma}\nabla^\mu K^{\nu\rho\sigma} - \frac{1}{3}\nabla_\mu K_{\nu\rho\sigma}\nabla^\mu K^{\rho\nu\sigma} + \nabla_\mu K^\alpha{}_{\alpha\nu}\nabla^\mu K_\beta{}^{\beta\nu}$$
$$+\frac{1}{3}\nabla_\mu K^\mu{}_{\rho\sigma}\nabla_\nu K^{\nu\rho\sigma} + \frac{2}{3}\nabla_\mu K^\mu{}_{\rho\sigma}\nabla_\nu K^{\rho\nu\sigma} + \frac{2}{3}\nabla_\mu K_\rho{}^\mu{}_\sigma\nabla_\nu K^{\rho\nu\sigma}$$
$$+\frac{1}{3}\nabla_\mu K_\rho{}^\mu{}_\sigma\nabla_\nu K^{\sigma\nu\rho} - \nabla_\mu K^\alpha{}_\alpha{}^\mu\nabla_\nu K^\beta{}_\beta{}^\nu + 2\nabla_\mu K^{\mu\nu}{}_\rho\nabla_\nu K^\alpha{}_\alpha{}^\rho\Big]$$

The Fourier transform of the linearized Lagrangian is

$$\mathcal{L} \;=\; -\frac{1}{2}K\cdot q^2\left[P_{22}\left(2^-\right) - P_{33}\left(1^-\right)\right]\cdot K$$

Spin 3^-

General MAG

$$
\begin{aligned}
\mathcal{L} \;=\; & -2b_1^{QQ}\big[\tfrac{1}{4}\nabla_\mu Q_{\alpha\beta\gamma}\nabla^\mu Q^{\alpha\beta\gamma} + \tfrac{1}{2}\nabla_\mu Q_{\alpha\beta\gamma}\nabla^\mu Q^{\beta\alpha\gamma} \\
& -\nabla_\mu \mathrm{tr}_{(12)}Q_\alpha \nabla^\mu \mathrm{tr}_{(12)}Q^\alpha - \nabla_\mu \mathrm{tr}_{(12)}Q_\alpha \nabla^\mu \mathrm{tr}_{(23)}Q^\alpha - \tfrac{1}{4}\nabla_\mu \mathrm{tr}_{(23)}Q_\alpha \nabla^\mu \mathrm{tr}_{(23)}Q^\alpha \\
& -\tfrac{1}{4}\mathrm{div}_{(1)}Q_{\alpha\beta}\mathrm{div}_{(1)}Q^{\alpha\beta} - \mathrm{div}_{(1)}Q_{\alpha\beta}\mathrm{div}_{(2)}Q^{\alpha\beta} - \mathrm{div}_{(2)}Q_{(\alpha\beta)}\mathrm{div}_{(2)}Q^{(\alpha\beta)} \\
& -\tfrac{1}{8}\mathrm{trdiv}_{(1)}Q_{\alpha\beta}\mathrm{trdiv}_{(1)}Q^{\alpha\beta} - \tfrac{1}{2}\mathrm{trdiv}_{(1)}Q_{\alpha\beta}\mathrm{trdiv}_{(2)}Q^{\alpha\beta} - \tfrac{1}{2}\mathrm{trdiv}_{(2)}Q_{\alpha\beta}\mathrm{trdiv}_{(2)}Q^{\alpha\beta} \\
& +\mathrm{div}_{(12)}Q_\alpha \mathrm{tr}_{(23)}Q^\alpha + \tfrac{1}{2}\mathrm{div}_{(23)}Q_\alpha \mathrm{tr}_{(23)}Q^\alpha \\
& +2\mathrm{div}_{(12)}Q_\alpha \mathrm{tr}_{(12)}Q^\alpha + \mathrm{div}_{(23)}Q_\alpha \mathrm{tr}_{(12)}Q^\alpha\big] \;,
\end{aligned}
$$

where $\mathrm{div}_{(1)}Q_{\alpha\beta} = \nabla^\lambda Q_{\lambda\alpha\beta}$ etc.
propagates only a massless spin-3 state.

Indeed setting $S_{\alpha\beta\gamma} = Q_{(\alpha\beta\gamma)}$, and $b_1^{QQ} = 1/3$ it becomes the Fronsdal Lagrangian

$$
\begin{aligned}
\mathcal{L} \;\; = \;\; & \frac{1}{2}\Big[- q^2 S_{\alpha\beta\gamma} S^{\alpha\beta\gamma} + 3q^2 \mathrm{tr}_{(12)} S_\alpha \mathrm{tr}_{(12)} S^\alpha \\
& + 3\mathrm{div}_{(1)} S_{\alpha\beta} \mathrm{div}_{(1)} S^{\alpha\beta} + \frac{3}{2}(\mathrm{trdiv}_{(1)} S)^2 - 6\mathrm{div}_{(12)} S_\alpha \mathrm{tr}_{(12)} S^\alpha \Big]
\end{aligned}
$$

The “higher spin symmetry” $\delta S_{\alpha\beta\gamma} = \partial_{(\alpha}\Lambda_{\beta\gamma)}$ is an accidental symmetry.

Massive case under investigation

(restrictions as an EFT - see e.g. B. Bellazzini, F. Riva, J. Serra and F. Sgarlata, JHEP **10** (2019), 189 [arXiv:1903.08664 [hep-th]])

Extension to the UV

In Landau-Ginzburg theory and in the SM the Goldstone bosons are embedded in a linear representation.
One more field is needed.

In gravity no new field needed. Simplest linear realization would keep the same fields but relax constraints.

In linear realization, unbroken phase can be realized,
but difficult to write actions that make sense in both phases.

Some quantum calculations in MAG

The RG for Palatini/Holst action
J.-E. Daum, M. Reuter, Phys.Lett.B 710 (2012) 215-218, e-Print: 1012.4280 [hep-th]

The RG for the general dimension-2 Lagrangian
C. Pagani and R.P. Class.Quant.Grav. 32 (2015) 19, 195019, e-Print: 1506.02882 [gr-qc]

The RG for non-integrable Weyl theory
C. Pagani and R.P. Class.Quant.Grav. 31 (2014) 115005, e-Print: 1312.7767 [hep-th]

All non-universal results.

Very recently

The beta functions generated by the Lagrangian

$$c_1 L_1^{FF} + a_1 L_1^{TT}$$

with

$$L_1^{FF} = F_{abcd} F^{abcd} \;, \qquad L_1^{TT} = T_{abc} T^{abc}$$

have been calculated by
O. Melichev, PhD thesis, SISSA 2023

In Einstein form

$$
\begin{aligned}
\mathcal{L} \;=\;& a_1 T_{\mu\nu\rho} T^{\mu\nu\rho} \\
&+c_1 \Big[R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + 5 R^{\mu\nu} \nabla^{\rho} T_{\rho\mu\nu} \\
&+\frac{3}{2} \nabla_{\alpha} T_{\mu\nu\rho} \nabla^{\alpha} T^{\mu\nu\rho} + \nabla_{\alpha} T_{\mu\nu\rho} \nabla^{\alpha} T^{\mu\rho\nu} \\
&+\nabla^{\alpha} T_{\alpha\mu\nu} \nabla_{\beta} (T^{\beta\mu\nu} + T^{\beta\nu\mu}) - \frac{1}{2} \nabla^{\alpha} T_{\mu\alpha\nu} \nabla_{\beta} T^{\mu\beta\nu} \\
&+2 \nabla^{\alpha} T_{\alpha\mu\nu} \nabla_{\beta} T^{\mu\beta\nu} \Big] \\
&+O(R\nabla T) + O(TT\nabla T) + O(TTTT) \,.
\end{aligned}
$$

A technical point

The Hessian contains

$$a_1 X_{\mu\nu}(-\bar{\nabla}^2 g^{\nu\sigma}+\bar{\nabla}^\sigma\bar{\nabla}^\nu)X^\mu{}_\sigma+c_1 Z_\mu{}^{\rho\sigma}(-\bar{\nabla}^2 g^{\mu\nu}+\bar{\nabla}^\nu\bar{\nabla}^\mu)Z_{\nu\rho\sigma}+\dots$$

Both in YM theory and in GR, such terms are removed by choosing Lorentz gauge ($\partial_\mu A^\mu$), resp. De Donder gauge with gauge parameter $\alpha=1$ (Feynman gauge).

If we use metric formalism, can eliminate one nonminimal term choosing the gauge $\bar{\nabla}_\mu X^\mu{}_\nu=0$, but not the other.

Gauge fixing

Working in tetrad formalism

$$\begin{aligned} S_{GF} &= \int d^4x\sqrt{|g|}\left[\frac{a_1}{\alpha_D}(\bar{\nabla}_\mu X^\mu{}_\nu)^2 + \frac{c_1}{\alpha_L}(\bar{\nabla}_\mu Z^{\mu ab})^2\right] \\ &= \int d^4x\sqrt{|g|}\Big[-a_1 X_{\mu\nu}\bar{\nabla}^\sigma\bar{\nabla}^\nu X^\mu{}_\sigma - c_1 Z_\mu{}^{\rho\sigma}\bar{\nabla}^\nu\bar{\nabla}^\mu Z_{\nu\rho\sigma} \\ &\quad + a_1 X_{\mu\nu}\bar{R}^{\nu\sigma}X^\mu{}_\sigma - a_1 X_{\mu\nu}\bar{R}^{\mu\rho\nu\sigma}X_{\rho\sigma} \\ &\quad + c_1 Z_{\mu\alpha\beta}\bar{R}^{\mu\sigma}Z_\sigma{}^{\alpha\beta} - 2c_1 Z_{\mu\alpha\beta}\bar{R}^{\mu\rho\alpha\sigma}Z_{\rho\sigma}{}^\beta\Big] \,. \end{aligned}$$

and choosing $\alpha_D = \alpha_L = 1$ eliminates all nonminimal terms.

Gauge fixed Hessian

Rescaling

$$X^a{}_\mu \to \frac{1}{\sqrt{a_1}} X^a{}_\mu \;, \qquad Z_\mu{}^a{}_b \to \frac{1}{\sqrt{c_1}} Z_\mu{}^a{}_b$$

and performing some integrations by parts,
one can write the gauge-fixed Hessian in the form

$$\begin{pmatrix} X & Z \end{pmatrix} \begin{pmatrix} -\nabla^2 + V^\mu_{XX}\nabla_\mu + W_{XX} & V^\mu_{XZ}\nabla_\mu + W_{XZ} \\ V^\mu_{ZX}\nabla_\mu + W_{ZX} & -\nabla^2 + V^\mu_{ZZ}\nabla_\mu + W_{ZZ} \end{pmatrix} \begin{pmatrix} X \\ Z \end{pmatrix}$$

Ghost action

$$\begin{aligned} S_{gh} &= \int d^4x\sqrt{\bar{g}}\left[\bar{\Sigma}(\delta_{\Sigma}^{QL}\chi_L + \tilde{\delta}_C^{QD}\chi_L) + \bar{C}(\delta_{\Sigma}^{QL}\chi_D + \tilde{\delta}_C^{QD}\chi_D)\right] \\ &= \int d^4x\sqrt{\bar{g}}\,(\bar{\Sigma} \quad \bar{C}) \begin{pmatrix} \Delta_{LL} & \Delta_{LD} \\ \Delta_{DL} & \Delta_{DD} \end{pmatrix} \begin{pmatrix} \Sigma \\ C \end{pmatrix} , \end{aligned}$$

$$\begin{aligned} (\Delta_{LL}\Sigma)^{\alpha}{}_{\beta} &= \bar{\nabla}^2\Sigma^{\alpha}{}_{\beta} + \bar{\nabla}^{\nu}(\bar{K}_{\nu}{}^{\alpha}{}_{\gamma}\Sigma^{\gamma}{}_{\beta} - \Sigma^{\alpha}{}_{\gamma}\bar{K}_{\nu}{}^{\gamma}{}_{\beta}) \\ (\Delta_{LD}C)^{\alpha}{}_{\beta} &= \bar{\nabla}^{\nu}(\bar{F}_{\rho\nu}{}^{\alpha}{}_{\beta}C^{\rho}) \\ (\Delta_{DL}\Sigma)^{\alpha} &= \bar{\nabla}^{\nu}\Sigma^{\alpha}{}_{\nu} \\ (\Delta_{DD}C)^{\alpha} &= \bar{\nabla}^2 C^{\alpha} + \bar{\nabla}^{\nu}(K_{\rho}{}^{\alpha}{}_{\nu}C^{\rho}) \end{aligned}$$

Log divergences

For an operator of the form $-\bar{\nabla}^2 + V^\mu \bar{\nabla}_\mu + W$
the logarithmically divergent part of $\mathrm{Tr}\log\mathcal{O}$ is

$$
\begin{aligned}
-\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\Big[&\tfrac{1}{180}\left(\bar{R}_{\mu\nu\rho\sigma}\bar{R}^{\mu\nu\rho\sigma} - \bar{R}_{\mu\nu}\bar{R}^{\mu\nu} + \tfrac{5}{2}\bar{R}^2\right)\mathrm{tr}\mathbb{I} \\
&+\tfrac{1}{2}\mathrm{tr}\,W^2 - \tfrac{1}{2}\mathrm{tr}\,W\bar{\nabla}_\mu V^\mu + \tfrac{1}{4}\mathrm{tr}\,WV_\mu V^\mu \\
&-\tfrac{1}{6}\bar{R}\,\mathrm{tr}\,W + \tfrac{1}{12}\bar{R}\,\mathrm{tr}\bar{\nabla}_\mu V^\mu - \tfrac{1}{24}\bar{R}\,\mathrm{tr}\,V_\mu V^\mu \\
&+\tfrac{1}{12}\mathrm{tr}\Omega_{\mu\nu}\Omega^{\mu\nu} - \tfrac{1}{6}\mathrm{tr}\,\Omega_{\mu\nu}\bar{\nabla}^\mu V^\nu + \tfrac{1}{24}\mathrm{tr}\,\Omega_{\mu\nu}[V^\mu, V^\nu] \\
&+\tfrac{1}{8}\mathrm{tr}\bar{\nabla}_\mu V^\mu\bar{\nabla}_\rho V^\rho - \tfrac{1}{8}\mathrm{tr}\bar{\nabla}_\mu V^\mu V_\rho V^\rho + \tfrac{1}{32}\mathrm{tr}\,V_\mu V^\mu V_\rho V^\rho \\
&+\tfrac{1}{24}\mathrm{tr}(\bar{\nabla}_\mu V_\nu - \bar{\nabla}_\nu V_\mu)\bar{\nabla}^\mu V^\nu - \tfrac{1}{24}\mathrm{tr}\,\bar{\nabla}_\mu V_\nu[V^\mu, V^\nu] \\
&+\tfrac{1}{192}\mathrm{tr}\,[V_\mu, V_\nu][V^\mu, V^\nu]\Big]\,.
\end{aligned}
$$

Log divergences in Einstein form

$$
\begin{aligned}
-\frac{1}{2}\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\Bigg[&\frac{239}{60}H_1^{RR}+\frac{103}{30}H_2^{RR}-\frac{7}{4}H_3^{RR}\\
&+\frac{299}{24}H_3^{RT}-\frac{13}{12}H_5^{RT}\\
&+\frac{9}{4}H_1^{TT}+2H_2^{TT}-2H_4^{TT}+\frac{7}{6}H_5^{TT}\\
&-\frac{35}{24}H_6^{TT}-\frac{15}{4}H_7^{TT}+3H_8^{TT}+\frac{7}{6}H_9^{TT}+\dots\Bigg]
\end{aligned}
$$

Log divergences in Cartan form

$$
\begin{aligned}
-\frac{1}{2}\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\Big[&\frac{76}{15}L_1^{FF}+\frac{107}{60}L_3^{FF}-\frac{86}{15}L_4^{FF}\\
&+\frac{137}{30}L_7^{FF}-\frac{17}{15}L_8^{FF}-\frac{7}{4}L_{16}^{FF}\\
&-\frac{91}{10}L_1^{FT}+\frac{13}{3}L_8^{FT}-\frac{93}{10}L_9^{FT}-\frac{39}{5}L_{13}^{FT}\\
&+\frac{19}{30}L_1^{TT}-\frac{1}{5}L_2^{TT}-\frac{47}{30}L_3^{TT}+\frac{32}{15}L_5^{TT}+\dots\Big]
\end{aligned}
$$

Lessons

A single dimension-4 term in the action generates all the others.

However, all the new terms can be removed by field redefinitions acting on $a_1 TT$. In Einstein form

$$\begin{aligned}
\delta T = \nabla\nabla T \quad &\rightarrow \quad \nabla T \nabla T \\
\delta T = \nabla R \quad &\rightarrow \quad R \nabla T \\
\delta T = RT \quad &\rightarrow \quad RTT \\
\delta T = T\nabla T \quad &\rightarrow \quad TT\nabla T \\
\delta T = TTT \quad &\rightarrow \quad TTTT
\end{aligned}$$

Summary

- MAG contains many modified theories of gravity (dark matter vs. modified gravity)
- gravitational Higgs phenomenon gives mass to the connection and explains why we have the Levi-Civita connection at low energy
- if some d.o.f. of the connection are light, MAG as an EFT is different from the theory of gravitons
- need understanding of propagating d.o.f.
- quantum behavior still largely unexplored