Remembrance of Things Past

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Based on arXiv:2110.08715 (Miao & Tsamis), 2302.04808 (Kasdagli & Ulloa) & 2302.11528 (Yesilyurt)

Inflationary Production of Scalars & Gravitons

- $ds^2 = -dt^2 + a^2(t) d\vec{x} \cdot d\vec{x}$ ➔ $H(t) = \frac{\dot{a}}{a}$, $\epsilon(t) = -\frac{\dot{H}}{H^2}$
  - Inflation is $H > 0$ & $0 \leq \epsilon < 1$ de Sitter ➔ $\epsilon = 0$, $\dot{H} = 0$, $a = e^{Ht}$
- Inflation produces LOT of infrared scalars & gravitons
  - $N_s\left(t, \vec{k}\right) = \frac{\pi \Delta_{\mathcal{R}}^2(k)}{4Gk^2} \times \epsilon(t) a^2(t)$, $N_h\left(t, \vec{k}\right) = \frac{\pi \Delta_h^2(k)}{32Gk^2} \times a^2(t)$
- Interactions of these particles induce large logarithms
- E.g. $\mathcal{L} = -\frac{1}{2} \partial_\mu \Phi \partial_\nu \Phi g^{\mu\nu} \sqrt{-g} - \frac{\lambda}{4!} \Phi^4$ on de Sitter
  - $\langle \Omega | T_{\mu\nu} | \Omega \rangle = p \times g_{\mu\nu} + (\rho + p) u_\mu u_\nu$
  - $\rho = \frac{H^4}{32\pi^2}\left\{-3 + \frac{\lambda}{4\pi^2}[\ln(a)]^2 + O\left(\lambda^2\right)\right\}$ $p = \frac{H^4}{32\pi^2}\left\{+3 - \frac{\lambda}{4\pi^2}\left[[\ln(a)]^2 + \frac{2}{3}\ln(a)\right] + O\left(\lambda^2\right)\right\}$
- "Leading Logarithm" at order λ^N is $[\ln(a)]^{2N}$, lower powers are "sub-leading"

Large Logs from Loops of Inflationary Gravitons

- Scalar effects are *simpler*, but graviton effects are *generic*
- Gravitons can also do things scalars cannot because of *spin*
  - Provides an interaction which does not redshift
- GR + EM (arXiv:1308.3453 & 1408.1448)
  - $\Phi(t,r) = \frac{Q}{4\pi a r}\left\{1 + \frac{2G}{3\pi a^2 r^2} + \frac{2GH^2}{\pi}\ln(aHr) + \cdots\right\}$
  - $F^{0i}(t,\vec{x}) = F_0^{0i}(t,\vec{x})\left\{1 + \frac{2GH^2}{\pi}\ln(a) + \cdots\right\}$
- Pure GR (arXiv:2107.13905 & 2206.11467)
  - $u(t,k) = u_0(t,k)\left\{1 + \frac{16GH^2}{3\pi}\ln(a)^2 + \cdots\right\}$
  - $\Psi(t,r) = -\frac{GM}{ar}\left\{1 + \frac{103G}{15\pi a^2 r^2} - \frac{8GH^2}{\pi}\left[\ln(a)^3 - 3\ln(a)\ln(Hr)\right] + \ldots\right\}$
- Perturbation theory breaks down for $GH^2 \times \ln(a)^{\#} \sim 1$
  - Evolve later by summing series of leading logarithms
  - Late time effects requires formalism for general $a(t)$ with primordial inflation

Starobinsky's Stochastic Formalism

- Works for scalar potential models
  - $\mathcal{L} = -\frac{1}{2}\partial_\mu \Phi \partial_\nu \Phi g^{\mu\nu}\sqrt{-g} - V(\Phi)\sqrt{-g}$
    - Recall $V(\Phi) = \frac{\lambda}{4!}\Phi^4$ ➔ $\rho(t) = \frac{H^4}{32\pi^2}\left\{-3 + \frac{\lambda}{4\pi^2}[\ln(a)]^2 + O\left(\lambda^2\right)\right\} \rightarrow \frac{3\Gamma(\frac{3}{4})H^4}{8\pi^2\Gamma(\frac{1}{4})}$
- Replace Heisenberg field equation for Φ with Langevin equation for φ
  - $\partial_\mu\left[\sqrt{-g}\, g^{\mu\nu}\partial_\nu \Phi\right] = \sqrt{-g}\, V'(\Phi)$ ➔ $3H(\dot{\varphi} - \dot{\varphi}_0) = -V'(\varphi)$
    - "Stochastic jitter" $\varphi_0(t,\vec{x}) \equiv \int \frac{d^3k}{(2\pi)^3}\theta(k-H)\theta(Ha-k)\left\{\frac{H}{\sqrt{2k^3}}e^{i\vec{k}\cdot\vec{x}}\alpha(\vec{k}) + \frac{H}{\sqrt{2k^3}}e^{-i\vec{k}\cdot\vec{x}}\alpha^\dagger(\vec{k})\right\}$
    - Correlators of $\varphi(t,\vec{x})$ produce the same leading logs as $\Phi(t,\vec{x})$ to ALL ORDERS
  - Can also get late time limit when one is approached ➔ $\rho(t) \rightarrow \frac{3\Gamma(\frac{3}{4})H^4}{8\pi^2\Gamma(\frac{1}{4})}$
- Integrate to Yang-Feldman Equation, IR truncate, then differentiate
  - $\Phi(t,\vec{x}) = \Phi_0(t,\vec{x}) - \int d^4x' \sqrt{-g(t',\vec{x}')}\, i\theta(t-t')[\Phi_0(t,\vec{x}), \Phi_0(t',\vec{x}')]\, V'(\Phi(t,\vec{x}))$
    - $\Phi_0(t,\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\left\{u(t,k)e^{i\vec{k}\cdot\vec{x}}\alpha(\vec{k}) + u^*(t,k)e^{-i\vec{k}\cdot\vec{x}}\alpha^\dagger(\vec{k})\right\}$
  - Every Φ_0 must contribute an IR log to reach leading logarithm ➔ can IR truncate Φ_0 to φ_0
    - Differentiating gives Starobinsky's Langevin equation!
- Derivative interactions prevent some Φ_0's from contributing an IR log
  - Fundamental interaction of GR is $\sqrt{16\pi G} \times h\partial h\partial h$

Applying it to Nonlinear Sigma Models on de Sitter

- Single Field Model (unit S-matrix but interesting background & kinematics)
  - $\mathcal{L} = -\frac{1}{2}\left(1 + \frac{\lambda}{2}\Phi\right)^2 \partial_\mu \Phi \partial_\nu \Phi g^{\mu\nu}\sqrt{-g}$
  - $\frac{\delta S}{\delta \Phi} = \left(1 + \frac{\lambda}{2}\Phi\right) \partial_\mu \left[\left(1 + \frac{\lambda}{2}\Phi\right)\sqrt{-g}\, g^{\mu\nu}\partial_\nu \Phi\right] = 0$
- Integrate out differentiated fields in constant background from interaction
  - $\Phi(x) = \Phi_0$ ➔ $\langle\Omega|\Phi(x)\Phi(x')|\Omega\rangle = \frac{i\Delta(x;x')}{\left(1+\frac{\lambda}{2}\Phi_0\right)^2}$
  - $-V'_{\rm eff}(\Phi_0)\sqrt{-g} \equiv \left(1 + \frac{\lambda}{2}\Phi_0\right) \partial_\mu \left[\frac{\lambda}{4}\sqrt{-g}\, g^{\mu\nu}\partial_\nu \langle\Omega|\Phi^2|\Omega\rangle\right] \rightarrow \frac{3\lambda H^4}{16\pi^2}\frac{\sqrt{-g}}{1+\frac{\lambda}{2}\Phi_0}$
- $V_{\rm eff}(\Phi) = \frac{3H^4}{8\pi^2}\ln\left|1 + \frac{\lambda}{2}\Phi\right|$ a scalar potential model! ➔ use Starobinsky
  - $\left(1 + \frac{\lambda}{2}\Phi\right) \partial_\mu \left[\left(1 + \frac{\lambda}{2}\Phi\right)\sqrt{-g}\, g^{\mu\nu}\partial_\nu \Phi\right] = -V'_{\rm eff}(\Phi)\sqrt{-g}$ ➔ $3H(\dot{\varphi} - \dot{\varphi}_0) = -\frac{V'_{\rm eff}(\varphi)}{\left(1+\frac{\lambda}{2}\varphi\right)^2}$
- VEV shows ``classical'' roll-down accelerated by stochastic jitter
  - $\langle\Omega|\Phi|\Omega\rangle = \frac{2}{\lambda}\left\{\left[1 - \frac{\lambda^2 H^2}{8\pi^2}\ln(a)\right]^{1/4} - 1\right\} - \frac{3\lambda^3 H^4}{2^8\pi^4}\ln(a)^2 + O(\lambda^5)$

Curvature-Dependent Renormalizations

- Some large logarithms come from the UV
  - (Primitive $= \frac{1}{D-4}$) $-$ (Counterterm $= \frac{a^{D-4}}{D-4}$) $= -\ln(a) + O(D-4)$
  - Doesn't happen for scalar potential models whose leading logs are UV finite
- UV logs are captured by the RG
- Two Field Model: $\mathcal{L} = -\frac{1}{2}\partial_\mu A \partial_\nu A g^{\mu\nu}\sqrt{-g} - \frac{1}{2}\left(1 + \frac{\lambda}{2}A\right)^2 \partial_\mu B \partial_\nu B g^{\mu\nu}\sqrt{-g}$
  - $\Delta\mathcal{L} = -\frac{1}{2}C_{B1}\square B \square B\sqrt{-g} - \frac{1}{2}C_{B2}R\partial_\mu B\partial_\nu B g^{\mu\nu}\sqrt{-g}$
  - The C_{B1} term intrinsically HD, but the C_{B2} term is $\delta Z_B = C_{B2}R$
  - $C_{B2} = \frac{\lambda^2\mu^{D-4}}{4(4\pi)^{D/2}}\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\frac{\pi\cot(\frac{D\pi}{2})}{D(D-1)} - \frac{\lambda^2\mu^{D-4}}{32\pi^{D/2}}\frac{\Gamma(\frac{D}{2}-1)}{2(D-3)(D-4)}\left(\frac{D-2}{D-1}\right)$
  - $\gamma_B \equiv \frac{\partial\ln(1+\delta Z_B)}{\partial\ln(\mu^2)} = -\frac{\lambda^2 H^2}{32\pi^2} + O(\lambda^4)$ and $\beta = O(\lambda^5)$
- Callan-Symanzik Equation
  - $\left[\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial\lambda} - 2\gamma_B\right]P_B(t,r) = 0$ and $P_B(t,r) \to \frac{KH}{4\pi}\ln(Hr) + O(\lambda^2)$
  - $\mu \to r$ ➔ $P_B(t,r) \to \frac{KH}{4\pi}\ln(Hr)\left\{1 - \frac{\lambda^2H^2}{32\pi^2}\ln(Hr) + O(\lambda^4)\right\}$

Large Logarithms in Nonlinear Sigma Models
Stochastic and Renormalization Group

Single Field Model

Quantity	Leading Logarithms
$u_\Phi(\eta,k)$	$\left\{1+\frac{\lambda^2H^2}{32\pi^2}\ln(a)+O(\lambda^4)\right\}\times\frac{H}{\sqrt{2k^3}}$
$P_\Phi(\eta,r)$	$\left\{1+\frac{\lambda^2H^2}{32\pi^2}\ln(a)+O(\lambda^4)\right\}\times\frac{KH}{4\pi}\ln(Hr)$
$\langle\Omega\vert\Phi(x)\vert\Omega\rangle$	$-\left\{1+\frac{15\lambda^2H^2}{64\pi^2}\ln(a)+O(\lambda^4)\right\}\times\frac{\lambda H^2}{16\pi^2}\ln(a)$
$\langle\Omega\vert\Phi^2(x)\vert\Omega\rangle_{\rm ren}$	$\left\{1+\frac{15\lambda^2H^2}{64\pi^2}\ln(a)+O(\lambda^4)\right\}\times\frac{H^2}{4\pi^2}\ln(a)$

Double Field Model

Quantity	Leading Logarithms
$u_A(\eta,k)$	$\left\{1-\frac{\lambda^2H^2}{32\pi^2}\ln(a)+O(\lambda^4)\right\}\times\frac{H}{\sqrt{2k^3}}$
$u_B(\eta,k)$	$\left\{1+0+O(\lambda^4)\right\}\times\frac{H}{\sqrt{2k^3}}$
$P_A(\eta,r)$	$\left\{1-\frac{\lambda^2H^2}{32\pi^2}\ln(a)+\frac{\lambda^2H^2}{32\pi^2}\ln(Hr)+O(\lambda^4)\right\}\times\frac{KH}{4\pi}\ln(Hr)$
$P_B(\eta,r)$	$\left\{1-\frac{\lambda^2H^2}{32\pi^2}\ln(Hr)+O(\lambda^4)\right\}\times\frac{KH}{4\pi}\ln(Hr)$
$\langle\Omega\vert A(x)\vert\Omega\rangle$	$\left\{1+\frac{\lambda^2H^2}{64\pi^2}\ln(a)+O(\lambda^4)\right\}\times\frac{\lambda H^2}{16\pi^2}\ln(a)$
$\langle\Omega\vert A^2(x)\vert\Omega\rangle_{\rm ren}$	$\left\{1-\frac{\lambda^2H^2}{64\pi^2}\ln(a)+O(\lambda^4)\right\}\times\frac{H^2}{4\pi^2}\ln(a)$
$\langle\Omega\vert B(x)\vert\Omega\rangle$	0
$\langle\Omega\vert B^2(x)\vert\Omega\rangle_{\rm ren}$	$\left\{1+\frac{3\lambda^2H^2}{32\pi^2}\ln(a)+O(\lambda^4)\right\}\times\frac{H^2}{4\pi^2}\ln(a)$

How It Works for Gravity

- Curvature-Dependent Induced Stress Tensor
  - Analog of constant scalar is constant $h_{\mu\nu}$ ➔ constant $\tilde{g}_{\mu\nu} \equiv \eta_{\mu\nu} + \kappa h_{\mu\nu}$
  - But $g_{\mu\nu} = a^2 \tilde{g}_{\mu\nu}$ with constant $\tilde{g}_{\mu\nu}$ is de Sitter with $H^2 \rightarrow -\tilde{g}^{00} H^2$!

$$\Gamma^{\rho}_{\mu\nu} = aH\left(\delta^{\rho}_{\mu}\delta^{0}_{\nu} + \delta^{\rho}_{\nu}\delta^{0}_{\mu} - \tilde{g}^{0\rho}\tilde{g}_{\mu\nu}\right) ➔ R^{\rho}_{\sigma\mu\nu} = -\tilde{g}^{00} H^2 \left(\delta^{\rho}_{\mu} g_{\sigma\nu} - \delta^{\rho}_{\nu} g_{\sigma\mu}\right)$$

  - Can use the same relations to integrate out differentiated fields!
  - E.g. $T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma}\partial_\rho\varphi\partial_\sigma\varphi \;\; \rightarrow \;\; \frac{3}{32\pi^2}\left[-\tilde{g}^{00}H^2\right]^2 g_{\mu\nu}$
  - NB a negative contribution to the cosmological constant & arbitrarily large
- Curvature-Dependent Renormalizations
  - 1-loop counterterms are $C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta}$ & R^2
  - $C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta}$ is fundamentally higher-derivative ➔ irrelevant for large logs
  - But R^2 induces curvature-dependent renormalizations of G & $\Lambda = (D-1)H^2$

$$R^2 = [R - D\Lambda]^2 + 2D\Lambda[R - (D-2)\Lambda] + D(D-4)\Lambda^2$$

Facilitating the Stochastic Formalism

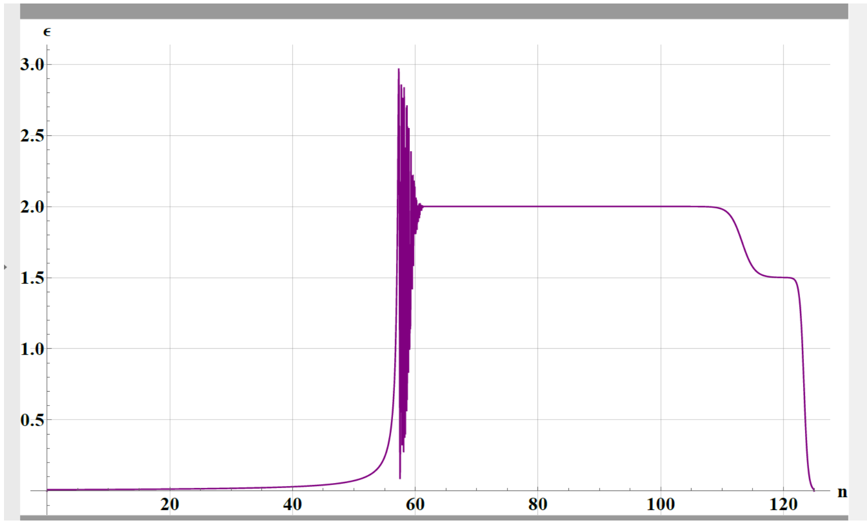
- Need coincident free propagator & 2 derivatives for cosmology
  - $\mathrm{A(t)} \equiv i\Delta(x;x)$, $B_\mu(t) \equiv \partial_\mu i\Delta(x;x')_{x'=x}$, $C_{\mu\nu}(t) \equiv \partial_\mu \partial'_\nu i\Delta(x;x')_{x'=x}$
  - $ds^2 = -dt^2 + a^2(t) d\vec{x} \cdot d\vec{x}$ (primordial inflation) $H(t) \equiv \frac{\dot{a}}{a}$, $\epsilon(t) \equiv -\frac{\dot{H}}{H^2}$
- Single scalar model ➔ $V_{\rm eff}(\Phi) = -\frac{1}{2} \Box A \times \ln\left(1 + \frac{\lambda}{2}\Phi\right)$
  - Can at least solve numerically provided $A(t)$ is known
- Get everything from $A(t)$
  - $B_\mu(t) = \frac{1}{2} \partial_\mu A(t)$
  - $g^{\mu\nu} C_{\mu\nu} = \frac{1}{2} \Box A$ & other independent component from conservation

Develop analytic approximations
Assuming primordial inflation

Typically three epochs

- "Inflation" $(0 < \epsilon < 1)$
  - Starts at t_i & stops at t_e
  - E-folding: $n \equiv \ln[\frac{a(t)}{a(t_i)}]$
- "Reheating"
  - $\epsilon(t)$ oscillates between 0 & 3
  - With increasing frequency
- "Hot Big Bang"
  - $\epsilon(t) = 2$ (Radiation domination)
  - $\epsilon(t) \cong 1.5$ (Matter domination)
  - $\epsilon(t) < 1$ (Late acceleration)

Typical history of $\epsilon(t)$ versus n



Fourier Mode Sum for $A(t)$ using $\mathcal{A}(t,k)$

- Mode function $u(t,k)$
  - $\ddot{u} + (D-1)H\dot{u} + \frac{k^2}{a^2}u = 0$, $u\dot{u}^* - \dot{u}u^* = \frac{i}{a^{D-1}}$
- Amplitude $\mathcal{A}(t,k) \equiv |u(t,k)|^2$ ➔ $A(t) = \int \frac{d^{D-1}k}{(2\pi)^{D-1}}\,\mathcal{A}(t,k)$
  - $\ddot{\mathcal{A}} - \frac{\dot{\mathcal{A}}^2}{2\mathcal{A}} + (D-1)H\dot{\mathcal{A}} + \frac{2k^2}{a^2}\mathcal{A} - \frac{1}{2a^{2D-2}\mathcal{A}} = 0$ with $\mathcal{A}(t,k) \to \frac{1}{2ka^{D-2}}$ for UV
- $\int \frac{d^D k}{(2\pi)^{D-1}} = \frac{2}{\Gamma(\frac{D-1}{2})(4\pi)^{(D-1)/2}} \int_{k_i}^{\infty} dk\, k^{D-2}$
  - $k_i \equiv a(t_i)H(t_i)$ ➔ initially super-horizon modes not amplified
- During Inflation: $\int_{k_i}^{\infty} dk k^{D-2}\mathcal{A} = \int_{aH}^{\infty} dk\,(UV) + \int_{k_i}^{aH} dk$ (1st crossing)
- After: $\int_{k_i}^{\infty} dk k^{D-2}\mathcal{A} = \int_{k_e}^{\infty} dk\,(UV) + \int_{aH}^{k_e} dk$ (2nd crossing) $+ \int_{k_i}^{aH} dk$ (1st crossing)
  - 1st crossing at t ➔ 2nd crossing at $t_2(t)$
  - 2nd crossing at t ➔ 1st crossing at $t_1(t)$

UV Expansion & 1st Crossing Form

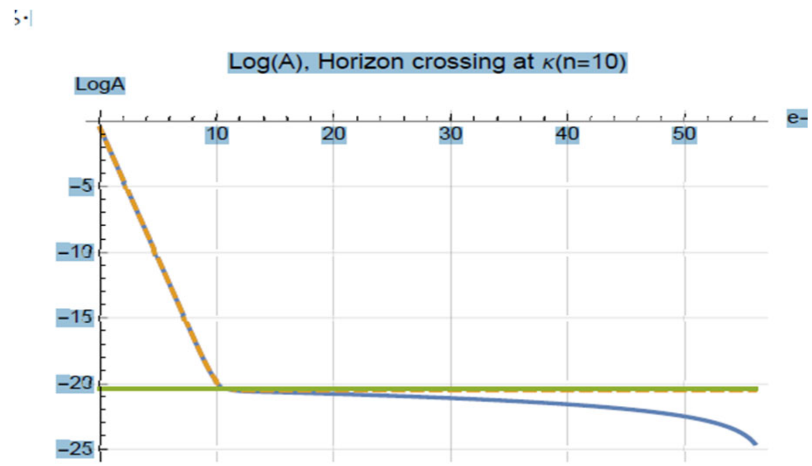
Analytic Forms are nearly perfect

& transition is sharp

Yellow = Numerical, Blue = UV expansion,

Green = 1st Crossing Form

- UV Expansion
  - $\mathcal{A} \to \frac{1}{2ka^{D-2}}\left\{1 + \frac{(D-2)(D-2\epsilon)}{8}\frac{a^2H^2}{k^2} + O\left(\frac{a^4H^4}{k^4}\right)\right\}$
- Form after 1st Crossing $\quad k \equiv a(t_k)H(t_k)$
  - $\mathcal{A} \to \frac{H^2(t_k)C(\epsilon(t_k))}{2k^3}$
  - $C(\epsilon) \equiv \frac{1}{\pi}\Gamma^2\left(\frac{1}{2} + \frac{1}{1-\epsilon}\right)[2(1-\epsilon)]^{\frac{1}{1-\epsilon}}$



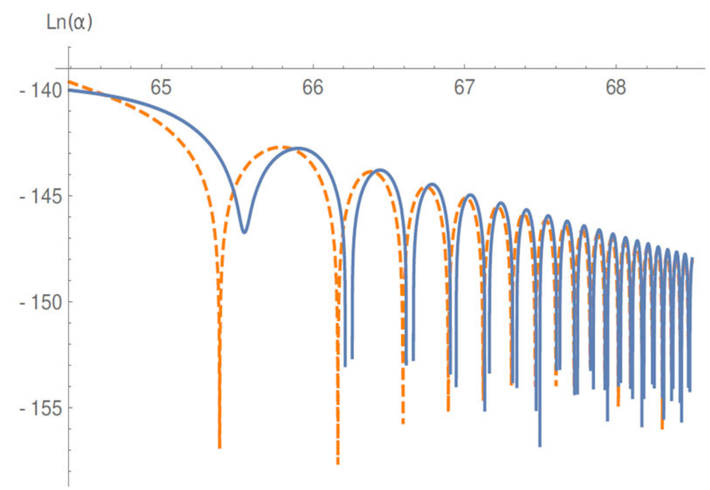
Form after 2nd Crossing

Analytic form

- $\mathcal{A} \to \frac{H^2(t_k)C(\epsilon(t_k))}{2k^3} \times \left[\frac{a(t_2(t))}{a(t)}\right]^2 \times \cos^2\left[\int_{t_2(t)}^{t} dt' \frac{k}{a(t')}\right]$
  - Comparison good except near $t_2(t_k)$
  - $\cos^2 \sim \frac{1}{2}$ inside an integral

Blue=Numerical

Yellow dashed = Analytic



Use dimensional regularization for the UV

- Recall $k^{D-2}\mathcal{A}$(t,k) $\cong \frac{k^{D-3}}{2a^{D-2}} + (D-2)(D-2\epsilon)H^2\frac{k^{D-5}}{8a^{D-4}}$
- $\int_{aH}^{\infty} dk k^{D-2}\mathcal{A}(t,k) \cong \left\{\frac{1}{2(D-2)}\left(\frac{k}{a}\right)^{D-2} + \frac{(D-2)(D-2\epsilon)H^2}{8(D-4)}\left(\frac{k}{a}\right)^{D-4}\right\}_{aH}^{\infty}$
  - $k \to \infty$ vanishes in dimensional regularization
  - Only lower limit contributes
- $\int_{aH}^{\infty} dk k^{D-2}\mathcal{A}(t,k) \cong -\left\{\frac{1}{2(D-2)} + \frac{(D-2)(D-2\epsilon)}{8(D-4)}\right\}H^{D-2}$
  - Renormalized by a conformal counterterm

Take $D = 4$ & convert $\int dk$ to $\int dt$ in finite parts

- $k = a(t_k)H(t_k)$ ➔ $\frac{dk}{k} = (1-\epsilon)Hdt$
- During inflation
  - $A(t) \cong -\frac{1}{8}\left(\frac{D-2}{D-4}\right)\frac{(D-2\epsilon)H^{D-2}}{\Gamma\left(\frac{D-1}{2}\right)(4\pi)^{\frac{D-1}{2}}} - \frac{H^2}{8\pi^2} + \frac{1}{4\pi^2}\int_{t_i}^{t} dt'\, H^3(t')[1-\epsilon(t')]\, C\big(\epsilon(t')\big)$
  - 1^{st} term $\equiv A_{\text{div}}(t)$ could be absorbed by conformal counterterm
- After inflation
  - $$A(t) \cong A_{\text{div}} - \frac{(2-\epsilon)H^2}{8\pi^2}\ln\left(\frac{k_e}{aH}\right) - \frac{1}{8\pi^2}\left(\frac{k_e}{a}\right)^2 + \frac{1}{8\pi^2 a^2}\int_{t_e}^{t} dt'\,[\epsilon(t')-1]H(t')a^2(t') \times H^2\big(t_1(t')\big)C\Big(\epsilon\big(t_1(t')\big)\Big) + \frac{1}{4\pi^2}\int_{t}^{t_2(t_i)} dt'\,[\epsilon(t')-1]H(t') \times H^2\big(t_1(t')\big)C\Big(\epsilon\big(t_1(t')\big)\Big)$$

Conclusions

- Graviton loops on de Sitter give factors of $\ln(a)$
  - We finally have a way of summing these up
  - And propagating them to arbitrarily late times
- Radiation domination ➔ $H(t) = \frac{1}{2t}$ & $a^2 H$ is constant
  - $A_{\rm fin}$ dominated by $\frac{1}{4\pi^2}\int_t^{t_2} dt' \, (\epsilon - 1)H(t') \times H^2(t_1) C\big(\epsilon(t_1)\big) \cong \ln(t) \times H_{\rm inf}^2$
- Large inflationary scales transmitted to late times
  - Generically $\square A_{\rm fin} \rightarrow H_{\rm inf}^2 \times H^2(t)$
- Lessons for building nonlocal models of cosmology
  - $\square A_{\rm fin}$ is some nonlocal scalar, but not simple
  - Form for general $a(t)$ enough for cosmology, but not gravitational force